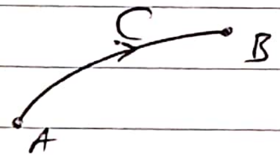
16.3 FTC FOR LINE INTEGRALS + CONSERVATIVE VFs

FTC 0

$$\int_a^b g'(t)\,dt = g(b) - g(a)$$

FTC I Let C be an oriented curve from A to B and let f be a function on $\mathbb{R}^n$



Then
$$\boxed{\int_C \nabla f \cdot d\vec{r} = f(B) - f(A)}$$

PROOF
$$\int_C \nabla f \cdot d\vec{r} = \int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$$
$$\overset{\text{CHAIN}}{\underset{\text{RULE}}{=}} \int_a^b (f \circ \vec{r})'(t)\,dt$$
$$\overset{\text{FTC 0}}{=} (f \circ \vec{r})(b) - (f \circ \vec{r})(a)$$
$$= f(B) - f(A)$$

DEF 1 A VF $\vec{F}$ on $\mathbb{R}^n$ is conservative if there is a function f so that

$$\vec{F} = \nabla f$$

NOTES

① We call f a POTENTIAL FUNCTION for $\vec{F}$

② f is analogous to an antiderivative for $\vec{F}$.

③ In Calc I every function F has an antiderivative given by

$$f(x) = \int_a^x F(t)\,dt$$

since $f' = F$ by FTC.

④ BUT there are vector fields $\vec{F}$ that are NOT conservative.

GOALS

I Why is it good for $\vec{F}$ to be conservative?
II How do you tell if $\vec{F}$ is conservative?
III If $\vec{F} = \nabla f$ how do you find f.?

I
NOTICE from FTC I: If $\vec{F} = \nabla f$ Then

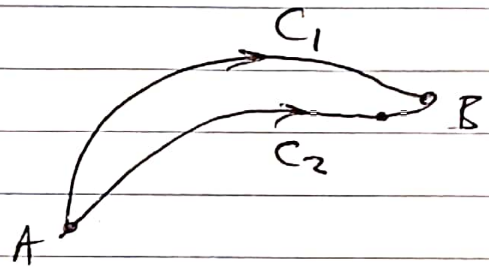
$$\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(B) - f(A)$$

only depends on values of f at endpts and NOT on path taken from A to B.

Def 2

We say $\int_C \vec{F}\cdot d\vec{r}$ is INDEPENDENT OF PATH

if
$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r}$$
whenever C_1 and C_2 have same starting and ending points



$\boxed{\text{THM I}}$ Suppose $\vec{F}$ is a VF on all of $\mathbb{R}^n$

Then

$\vec{F}$ is CONSERVATIVE $\Leftrightarrow$ $\int_C \vec{F}\cdot d\vec{r}$ is INDEPT OF PATH

PROOF

$\boxed{\Rightarrow}$ This follows from FTCI as explained above

$\boxed{\Leftarrow}$ FIX $(x_0, y_0) \in \mathbb{R}^2$. For any $(x,y) \in \mathbb{R}^2$ define
$$f(x,y) := \int_C \vec{F}\cdot d\vec{r}$$
where C is ANY Path from (x_0, y_0) to (x,y)

Since $\int_C \vec{F}\cdot d\vec{r}$ is assumed to be PATH INDEPT it doesn't matter which curve C we choose.

Picking special curves C one can show that

$$\nabla f = \vec{F} \text{ holds.}$$

So $\vec{F}$ is conservative $\square$

IMPORTANT EXAMPLE

Let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ be POSITION VF on $\mathbb{R}^3$
and $r = |\vec{r}|$

GRAVITATIONAL FORCE FIELD

Let $\vec{F}$ = Force on mass m at (x,y,z) due to mass M at $(0,0,0)$.

We know

- $|\vec{F}| = \dfrac{GmM}{r^2}$ INVERSE SQ LAW
- $\vec{F}$ points towards origin

So

$$\vec{F} = -GmM\frac{\vec{r}}{|\vec{r}|^3} = -GmM\frac{x\vec{i}+y\vec{j}+z\vec{k}}{(x^2+y^2+z^2)^{3/2}}$$

CLAIM Let $P = \frac{-GM}{r} = \frac{-GM}{\sqrt{x^2+y^2+z^2}}$

As $r \uparrow$ $f \uparrow$ to 0

be the GRAVITATIONAL POTENTIAL.

Then $\vec{F} = -m \nabla P = \nabla f$ for $f = -mP$.

So $\vec{F}$ is CONSERVATIVE !

REASON $\frac{\partial}{\partial x}(x^2+y^2+z^2)^{-1/2} = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}\,2x$

$$= \frac{-x}{(x^2+y^2+z^2)^{3/2}} \quad \text{ETC}$$

NOTE The Electric Force field is also conservative.

PHYSICS INTERPRETATION OF THM I

In a gravitational or electric force field $\vec{F}$ the work done to move a particle from A to B only depends on A, B and NOT on path between these two points.

II How do we tell if $\vec{F}$ is Conservative?
CAN'T USE THM I !!

THM II Let $\vec{F} = P\vec{i} + Q\vec{j}$ be a VF defined on all of $\mathbb{R}^2$. Then

$$\vec{F} \text{ is Conservative} \iff \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$$

PROOF

$\Rightarrow$ If $\vec{F} = P\vec{i} + Q\vec{j} = \nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}$

Then $P = \frac{\partial f}{\partial x}$, $Q = \frac{\partial f}{\partial y}$

So $\frac{\partial Q}{\partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial P}{\partial y}$

$\Leftarrow$ Uses Green's Thm (See Next Lecture)

III

METHOD to FIND f : $\vec{F} = \nabla f$

$$\vec{F} = 2x\sin y\,\vec{i} + (x^2\cos y - 3y^2)\vec{j}$$
$$= P\vec{i} + Q\vec{j}.$$

defined on all of $\mathbb{R}^3$

UES ✓ $\quad \frac{\partial P}{\partial y} = 2x\cos y = \frac{\partial Q}{\partial x}$.

So by Thm II we know $\vec{F} = \nabla f$ for some f

To find f: Know

$$\frac{\partial f}{\partial x} = P = 2x\sin y \quad (1)$$

$$\frac{\partial f}{\partial y} = Q = x^2\cos y - 3y^2 \quad (2)$$

So by (1) $\quad f(x,y) = \int 2x\sin y\,dx + g(y)$
$$= x^2\sin y + g(y) \quad (3)$$

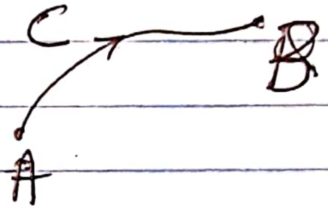
By (2) $\quad f(x,y) = \int (x^2\cos y - 3y^2)\,dy + h(x)$
$$= x^2\sin y - y^3 + h(x) \quad (4)$$

Compare (3) + (4) $\quad \boxed{f(x,y) = x^2\sin y - y^3}$

So $\int_C \vec{F}\cdot d\vec{r} = f(5,1) - f(-1,0) = 25\sin(1) - 1$.

C ANY CURVE FROM $(-1,0)$ to $(5,1)$

CONSERVATION OF ENERGY



$\vec{F}$ Force Field.

$\vec{r}(t) =$ Particle Trajectory.

① Newton's 2nd Law : $\vec{F}(\vec{r}(t)) = m\,\vec{r}\,''(t)$.

Work done by $\vec{F}$ a particle

$$= W = \int_C \vec{F} \cdot d\vec{r} = \int_a^b m\,\vec{r}\,''(t) \cdot \vec{r}\,'(t)\,dt$$

$$= \frac{m}{2} \int_a^b \frac{d}{dt}\left(\vec{r}\,'(t) \cdot \vec{r}\,'(t)\right) dt$$

$$= \frac{m}{2} \int_a^b \frac{d}{dt} \|\vec{r}\,'(t)\|^2\,dt$$

$$\overset{FTC}{=} \frac{m}{2}\|\vec{r}\,'(b)\|^2 - \frac{m}{2}\|\vec{r}\,'(a)\|^2$$

= Kinetic Energy ~~(KE)~~ at B − KE at A

$= K(B) - K(A)$

② IF $\vec{F} = \nabla f$ is conservative Then

$= -\nabla P$ (P = Potential in Physics)

So $W = -\int_C \nabla P \cdot d\vec{r} = -(P(B) - P(A)) = P(A) - P(B)$

③ Compare ①, ② :

$$K(A) + P(A) = K(B) + P(B)$$

* TOTAL ENERGY IS CONSERVED IN A CONSERVATIVE VF