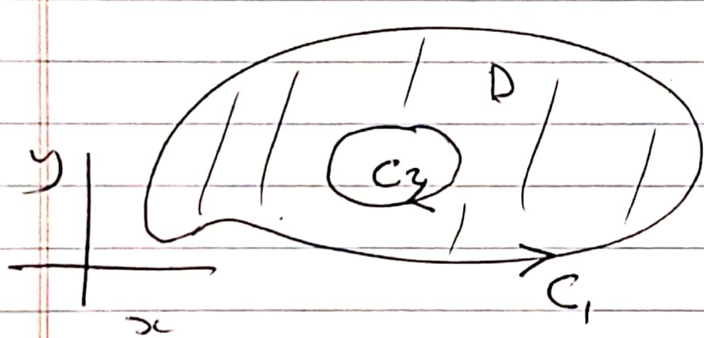
16.4 GREEN'S THEOREM

GREEN'S THM $\boxed{\text{FTC II}}$

Let D be a domain in $\mathbb{R}^2$ with boundary, ∂D.
Orient ∂D so that as you walk around ∂D with head up ($+\vec{k}$ direction) you keep D on your left.

D CAN HAVE HOLES

$\partial D = C_1 + C_2$



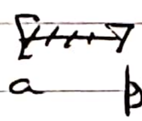
Let $\vec{F} = P\vec{i} + Q\vec{j}$ be a VF on D
Then

$$\boxed{\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_{\partial D} P\,dx + Q\,dy}$$

Green's Thm is FTC for Double Integrals

It is analogous to

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$



FTC on $\mathbb{R}$	Green's Thm
1D interval $[a,b]$	2D domain D
0D boundary $\{a\}\cup\{b\}$	1D boundary ∂D
Function F	Vector field $\vec{F} = P\vec{i} + Q\vec{j}$
Derivative F'	Derivative of $\vec{F}$ $= \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$

In both cases:

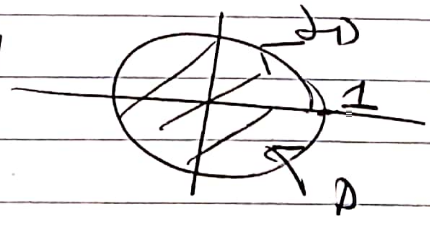
Integral of Derivative over Domain

= Integral (or Sum) of original vector field (or function) over boundary.

EX Lets verify Green's Thm for VF

$$\vec{F}(x,y) = -y\vec{i} + x\vec{j} \quad \text{on unit disc, } D$$

$\partial D: \ \vec{r}(t) = (\cos t, \sin t)$



$\boxed{\text{RHS}}$ $\int_C P\,dx + Q\,dy$

$$= \int_C -y\,dx + x\,dy = \int_0^{2\pi} (-\sin t)(-\sin t) + (\cos t)(\cos t)\,dt$$

$$= \int_0^{2\pi} 1\,dt = 2\pi$$

$\boxed{\text{LHS}}$ $$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= \iint_D \left[\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y)\right] dA$$

$$= \iint_D 2\, dA = 2.\text{Area}(D) = 2\pi$$

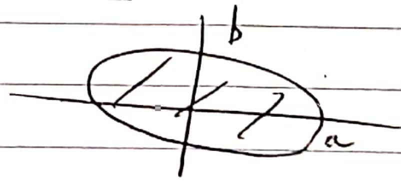
So LHS = RHS ✓✓

OBSERVATION

The calculations above show that for ANY DOMAIN

$$\boxed{\text{Area}(D) = \frac{1}{2}\int_{\partial D} -y\,dx + x\,dy}$$

Ex If D is the ellipse



Then ∂D has parametrization

$\vec{r}(t) = (a\cos t, \ b\sin t)$ $\quad 0 \le t \le 2\pi$

So $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$ ✓✓

THEN

$$\text{AREA}(D) = \frac{1}{2}\int_{\partial D} -y\,dx + x\,dy$$

$$= \frac{1}{2}\int_0^{2\pi} (-b\sin t)(-a\sin t) + (a\cos t)(b\cos t)\,dt$$

$$= \frac{1}{2}\int_0^{2\pi} ab\,dt$$

$$= \frac{ab}{\pi}\cdot 2\pi = \pi ab.$$

THIS GIVES YET ANOTHER PROOF OF FACT THAT AREA OF UNIT DISC $= \pi$ where 2π is defined to be circumference of unit circle!

Recall from 16.3

THM If $\vec{F} = P\vec{i} + Q\vec{j}$ is defined on all of $\mathbb{R}^2$

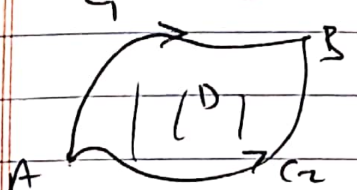
Then $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y} \Rightarrow \vec{F} = \nabla f$

~~We know $\int \vec{F}\cdot d\vec{r}$ Indep of Path $\Rightarrow \vec{F} = \nabla f$~~

PROOF

TO Show $\int_C \vec{F}\cdot d\vec{r}$ is Indep of Path

IF



$\partial D = C_2 - C_1$

Then $\int_{C_2} \vec{F}\cdot d\vec{r} - \int_{C_1} \vec{F}\cdot d\vec{r} = \int_{\partial D} \vec{F}\cdot d\vec{r}$

$$= \int_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_D 0\,dA = 0$$