16.5 CURL + DIVERGENCE

OVERVIEW

- CURL($\vec{F}$) AND DIV($\vec{F}$) are 2 ways to differentiate VFs.
- They give info about how fluids rotate (CURL) and expand/contract (DIV).
- 2 MORE VERSIONS OF FTC:
  - Stokes Thm: About CURL($\vec{F}$)
  - Div (Gauss') Thm: About DIV($\vec{F}$)

PDEs

- Maxwell's eqns that describe the time evolution of coupled $\vec{E}$, $\vec{B}$ fields (LIGHT) involve DIV and CURL
- Other equations involving DIV/CURL:
  - NAVIER STOKES eqns for fluid motion
  - Governing eqns for Continuum Mechanics.
    - NAIVERS eqn for linear elasticity.
  - The Heat equation.
  - Weather + Climate Modeling/Prediction.

THE DEL OPERATOR

$$\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}.$$

"Vector" whose components are $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$

EX **GRADIENT** of f:

$$\nabla f = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) f$$

$$= \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}.$$

CURL OF VF $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$

$$\mathrm{CURL}(\vec{F}) = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) - \vec{j}\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + \vec{k}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

So $\boxed{\nabla \times \vec{F} \text{ is a VECTOR FIELD}}$

EX $\vec{F} = xy\vec{i} + yz\vec{j} + zx\vec{k}$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & yz & zx \end{vmatrix} = \vec{i}\left(\frac{\partial}{\partial y}(zx) - \frac{\partial}{\partial z}(yz)\right) - \vec{j}\left(\frac{\partial}{\partial x}(zx) - \frac{\partial}{\partial z}(xy)\right) + \vec{k}\left(\frac{\partial}{\partial x}(yz) - \frac{\partial}{\partial y}(xy)\right)$$

$$= -y\vec{i} - z\vec{j} - x\vec{k}$$

THM

1. $\nabla \times \nabla f = \vec{0}$

2. Let $\vec{F}$ be a VF on $\mathbb{R}^3$, Then

$\vec{F}$ is conservative $\iff$ $\nabla \times \vec{F} = \vec{0}$.

PF

1. $$\nabla \times \nabla f = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$= \left(\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x} \right) \vec{i} + \cdots = \vec{0}$$

by Equality of Mixed Partials

2. LIKE Pf of Thm II of 16.3 except use Stokes' Thm instead of Green's Thm

$\boxed{\text{CURL ON } \mathbb{R}^2}$

IF $\vec{F}(x,y) = P(x,y)\vec{i} + Q(x,y)\vec{j}$ is a VF on $\mathbb{R}^2$

THEN $\boxed{\text{VB}}$ ✓

$$\nabla \times \vec{F} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}.$$

So Green's Thm

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_{\partial D} P\,dx + Q\,dy$$

becomes

$$\iint_D (\nabla \times \vec{F}) \cdot \vec{k}\, dA = \int_{\partial D} \vec{F} \cdot d\vec{r}$$

which is a special case of Stokes' Thm (16.8)

$\boxed{\text{DIVERGENCE}}$

$$\text{DIV}(\vec{F}) = \nabla \cdot \vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot \left(P\vec{i} + Q\vec{j} + R\vec{k}\right)$$

$$= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

$\boxed{\text{DIV}(\vec{F}) \text{ IS A FUNCTION}}$

EX $\nabla\cdot(xy\vec{i} + yz\vec{j} + zx\vec{k})$

$$= \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(yz) + \frac{\partial}{\partial z}(zx) = y + z + x.$$

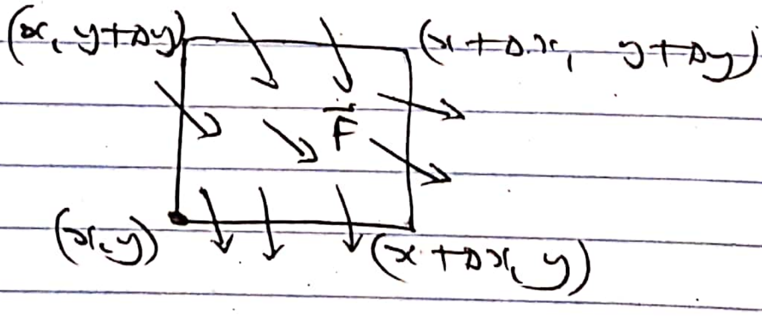
PHYSICAL MEANING OF DIV

Suppose $\vec{F}(x,y) = P(x,y)\vec{i} + Q(x,y)\vec{j}$ is a VF on $\mathbb{R}^2$

that is the $\boxed{\text{MOMENTUM DENSITY}}$ VF of a FLUID flowing in $\mathbb{R}^2$.

$$\vec{F} = \underbrace{\rho}_{\text{MASS DENSITY}}\ \underbrace{\vec{v}}_{\text{VELOCITY}}. \qquad \text{UNITS} \quad \frac{kg}{m^2}\cdot m/s.$$

SMALL RECTANGLE, R.



THM A

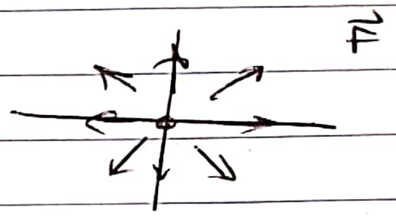
$$(\mathrm{DIV}\ \vec{F})(x,y) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \qquad \text{UNITS} \quad \frac{kg\,m}{m^2 s}\cdot\frac{1}{m} = \frac{kg}{m^2 s}$$

$$= \lim_{\text{Area}(R)\to 0} \frac{\text{Rate at which mass exits } R}{\text{Area of } R} \qquad \text{UNITS} \quad \frac{kg/s}{m^2}$$

= ~~MOMENTUM~~ MASS FLUX DENSITY AT (x,y).

EXS

1. POSITION VF on $\mathbb{R}^2$

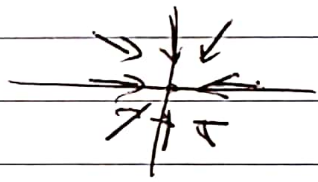


$$\vec{F} = x\vec{i} + y\vec{j}$$

$$\mathrm{DIV}(\vec{F}) = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) = 1 + 1 = 2 > 0$$

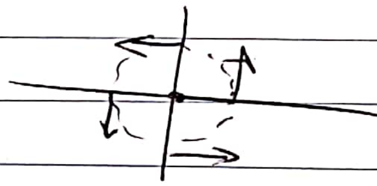
ON AVERAGE FLUID ~~LEA~~ EXITS SMALL DISC CENTERED AT $(0,0)$.

2. $\vec{F} = -x\vec{i} - y\vec{j}$ has $\mathrm{DIV}(\vec{F}) = -2 < 0$



On average fluid enters small disc center $(0,0)$

3. Merry - Go - Round VF

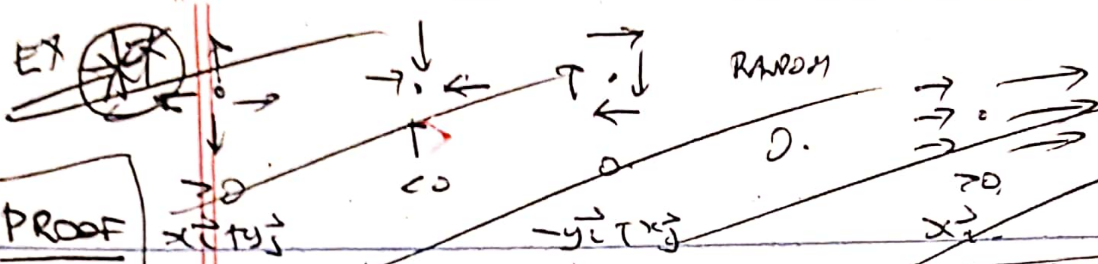


$$\vec{F} = -y\vec{i} + x\vec{j}$$

$$\mathrm{DIV}(\vec{F}) = \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x) = 0.$$

On average fluid neither exits/enters disc

Say fluid is INCOMPRESSIBLE



PROOF

If $\vec{F}$ is constant along bottom edge, Then

Mass crossing bottom edge per second out of R

$= $ Shaded Area

$= $ (Scalar Cpt of $\vec{F}$ in dirn of outward normal to edge) $\times$ (Length of Edge)

$$= \vec{F}\cdot(-\vec{j})\,\Delta x$$

$$= -Q(x,y)\,\Delta x \qquad \text{UNITS } \frac{kg}{ms}\,m = \frac{kg}{s} \checkmark$$

$\boxed{\text{EXIT RATES}}$

TOP $\vec{F}(x, y+\Delta y)\cdot\vec{j}\,\Delta x = Q(x, y+\Delta y)\,\Delta x$

BOT $\vec{F}(x,y)\cdot(-\vec{j})\,\Delta x = -Q(x,y)\,\Delta x$

RIGHT $\vec{F}(x+\Delta x, y)\cdot\vec{i}\,\Delta y = P(x+\Delta x, y)\,\Delta y$

LEFT $\vec{F}(x,y)\cdot(-\vec{i}\,\Delta y) = -P(x,y)\,\Delta y$

So Rate at which mass exits R

$$= [Q(x,y+\Delta y) - Q(x,y)]\,\Delta x + [P(x+\Delta x,y) - P(x,y)]\,\Delta y$$

$$\simeq \frac{\partial Q}{\partial y}\Delta y\,\Delta x + \frac{\partial P}{\partial x}\Delta x\,\Delta y$$

Since Area (R) $= \Delta x\,\Delta y$, MOMENTUM FLUX DENSITY

$$= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \quad \text{HOLDS} \quad \square$$

PHYSICAL MEANING OF $\nabla \times \vec{F}$

THM Suppose for today that $\vec{F} = P\vec{i} + Q\vec{j}$ is a VF in $\mathbb{R}^2$.

Then $\nabla \times \vec{F} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$

(a) IF $\nabla \times \vec{F} = \vec{0}$ at (x_0, y_0) Then on average fluid does NOT rotate about (x_0, y_0)

(b) IF $\nabla \times \vec{F}$ pts in dirn $+\vec{k}$ at (x_0, y_0) then (by RH Rule), on average fluid rotates CCW about (x_0, y_0).

(c) SIMILARLY: $-\vec{k} \longleftrightarrow$ CW

PROOF Uses Stokes' Thm. See Next Time.

EXS

(a) $\vec{F} = x\vec{i} + y\vec{j}$ $\quad \nabla \times \vec{F} = \vec{0}$.

(b) $\vec{F} = -y\vec{i} + x\vec{j}$ $\quad \nabla \times \vec{F} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k} = 2\vec{k}$ CCW ✓