16.6 PARAMETRIC SURFACES

①

<u>RECALL</u> A parametrization of a <u>curve</u> is a vector-valued function $\vec{r}(t)$ of a single variable t,

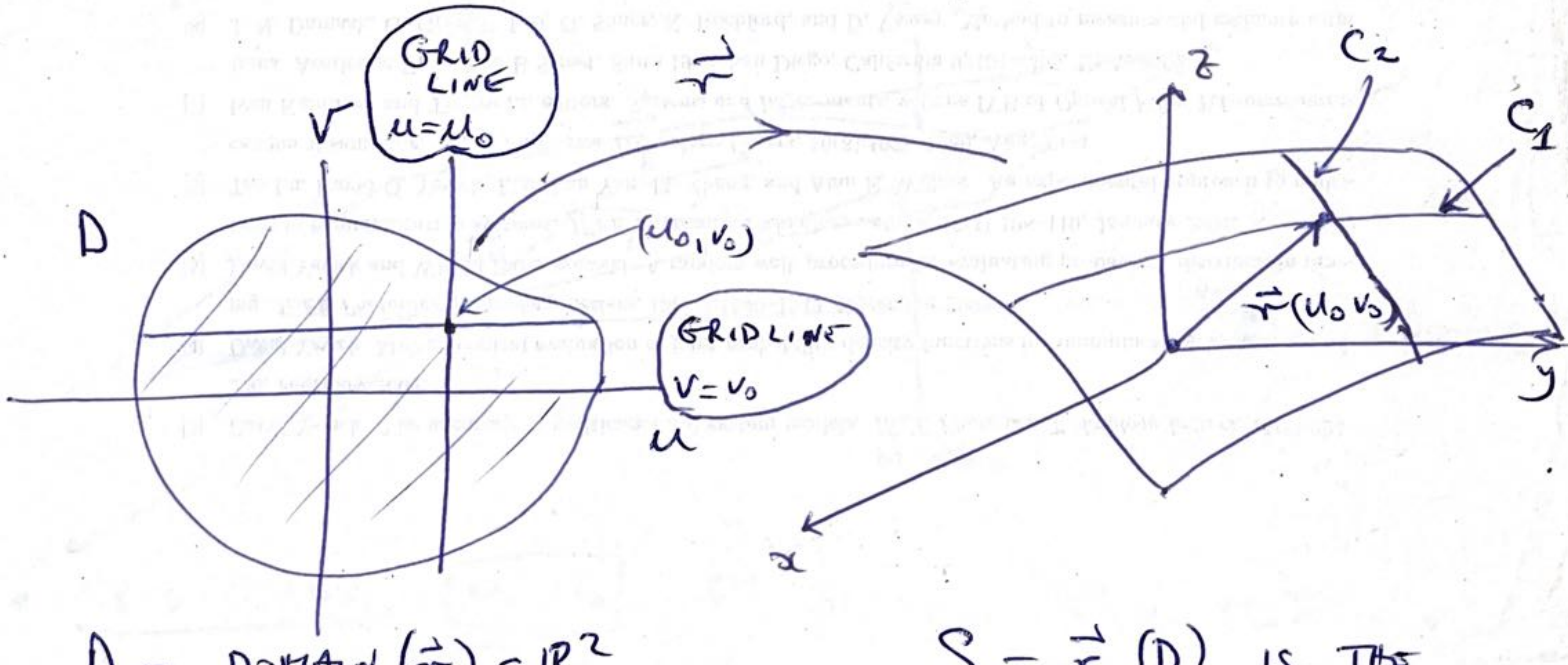
$$\vec{r} : \mathbb{R} \longrightarrow \mathbb{R}^3$$

<u>EX</u> $\vec{r}(t) = (\cos t, \sin t, t^2)$

<u>DEFN</u> A <u>parametrization of a surface</u> is a vector-valued function $\vec{r}(u,v)$ of 2 variables u, v

$$\vec{r} : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$

$$\boxed{\vec{r}(u,v) = x(u,v)\,\vec{i} + y(u,v)\,\vec{j} + z(u,v)\,\vec{k}}$$



$D = \text{DOMAIN}(\vec{r}) \subset \mathbb{R}^2$ $S = \vec{r}(D)$ IS THE <u>SURFACE</u>

$\underline{C_1}$ is the grid curve where $u = u_0$

It is parametrized by

$$\vec{r}(v) = \vec{r}(u_0, v).$$

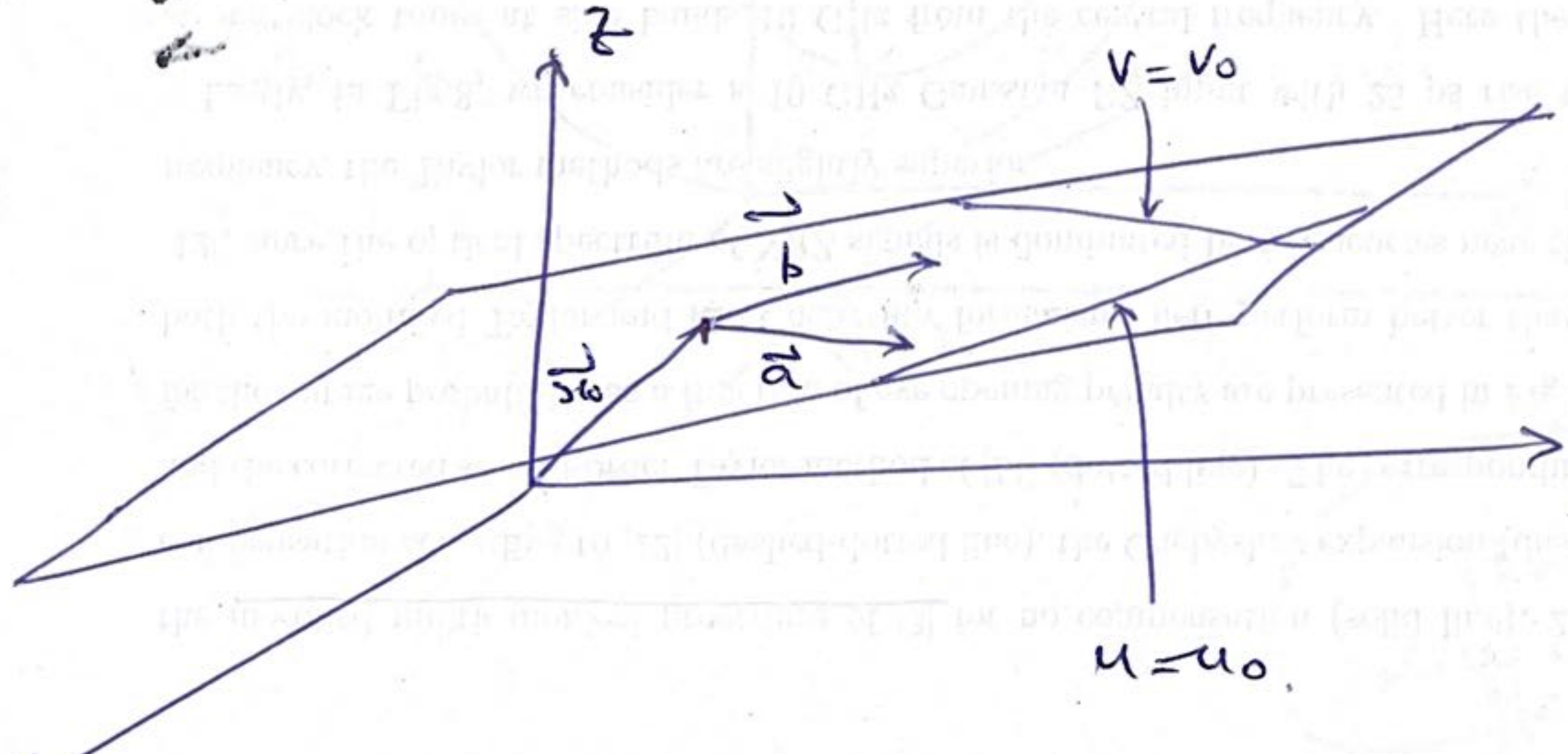
EXS

1 PLANES

As in #13,

$$\vec{r}(u,v) = \vec{x}_0 + u\vec{a} + v\vec{b}$$

is the parametrization of plane thru $\vec{x}_0$ containing vectors $\vec{a}, \vec{b}$.



• GRID CURVES $\boxed{u = u_0}$ $\vec{r}(u_0, v) = \vec{x}_0 + u_0\vec{a} + v\vec{b}$

$\boxed{v = v_0}$ $\vec{r}(u, v_0) = \vec{x}_0 + u\vec{a} + v_0\vec{b}$

② GRAPHS OF FUNCTIONS $z = f(x,y)$

$x = u$

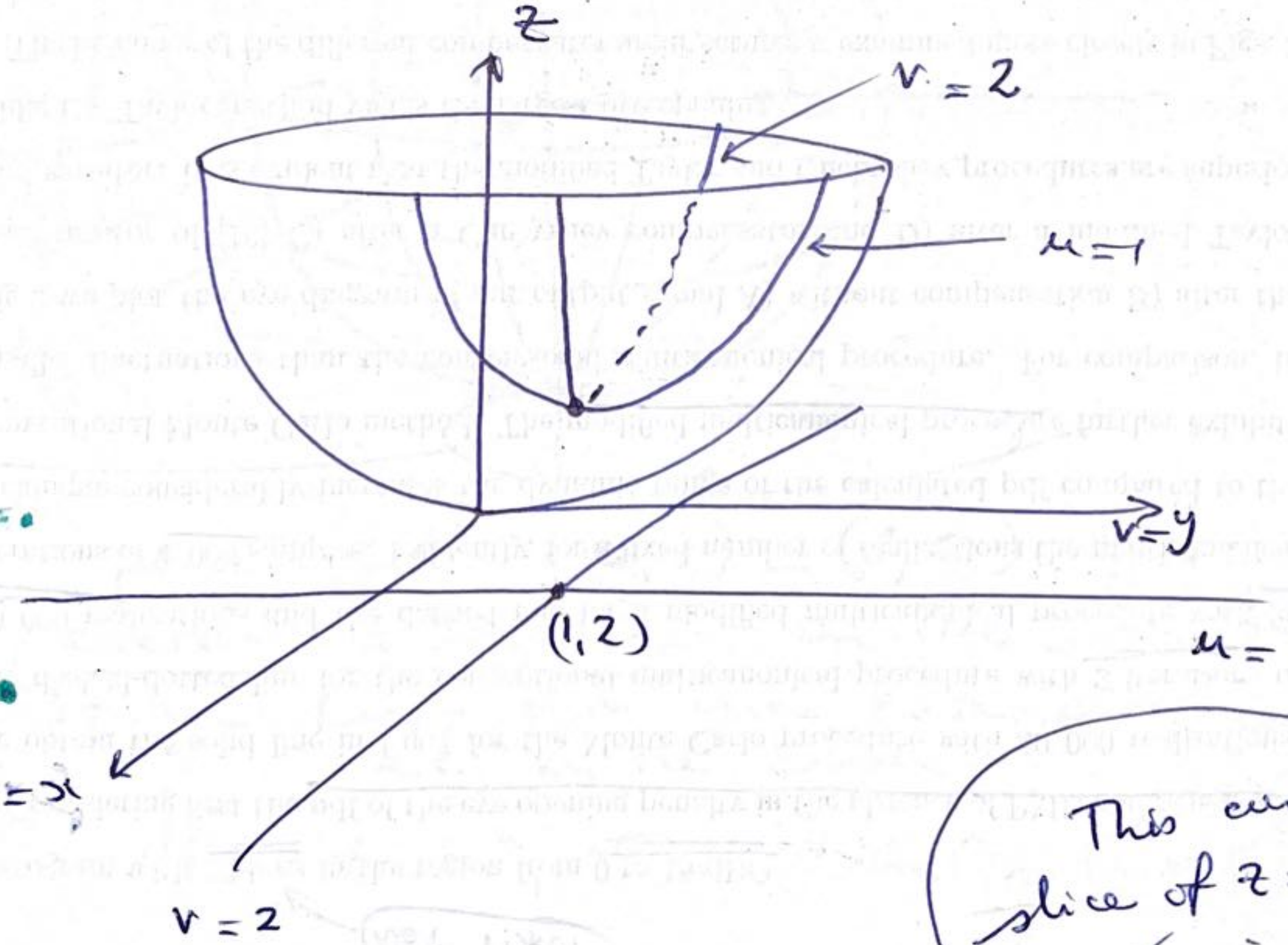
$y = v$

$z = f(u,v)$

i.e. $\vec{r}(u,v) = (u, v, f(u,v))$

EX $z = f(x,y) = 3x^2 + 4y^2$.

$\vec{r}(u,v) = (u, v, 3u^2 + 4v^2)$



This curve is slice of $z = f(x,y)$ in plane $x=1$

GRID CURVES THRU $(u_0, v_0) = (1,2)$

$\boxed{u = u_0 = 1}$ $\vec{r}(v) = \vec{r}(1,v) = (1, v, 3 + 4v^2)$

$\boxed{v = v_0 = 2}$ $\vec{r}(u) = \vec{r}(u,2) = (u, 2, 3u^2 + 16)$

3 SPHERE $x^2+y^2+z^2=R^2$ ⊛

Use spherical coordinates $(u,v)=(\theta,\phi)$

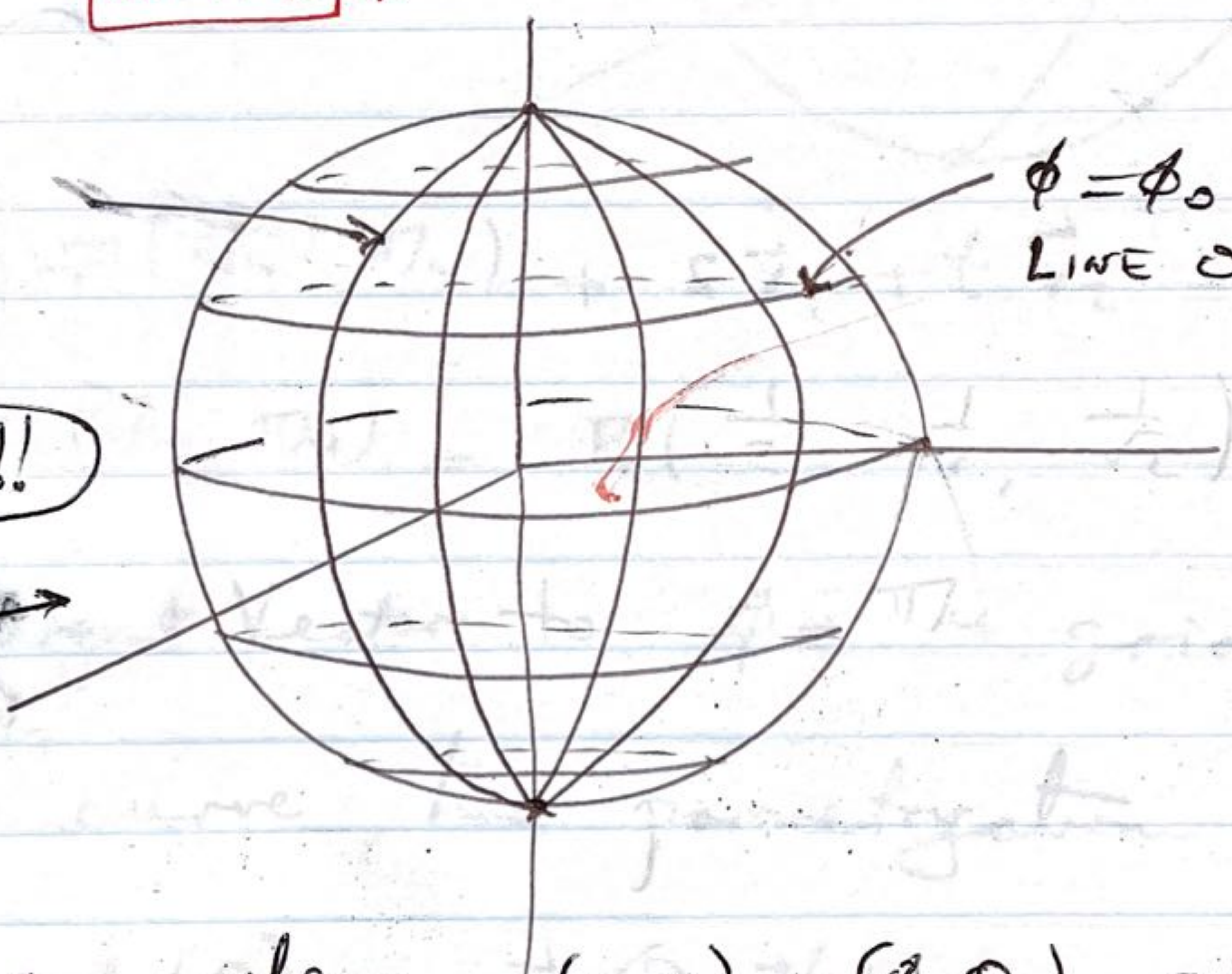
$$x = R\sin\phi\cos\theta$$
$$y = R\sin\phi\sin\theta \qquad 0<\theta<2\pi$$
$$z = R\cos\phi \qquad 0<\phi<\pi$$

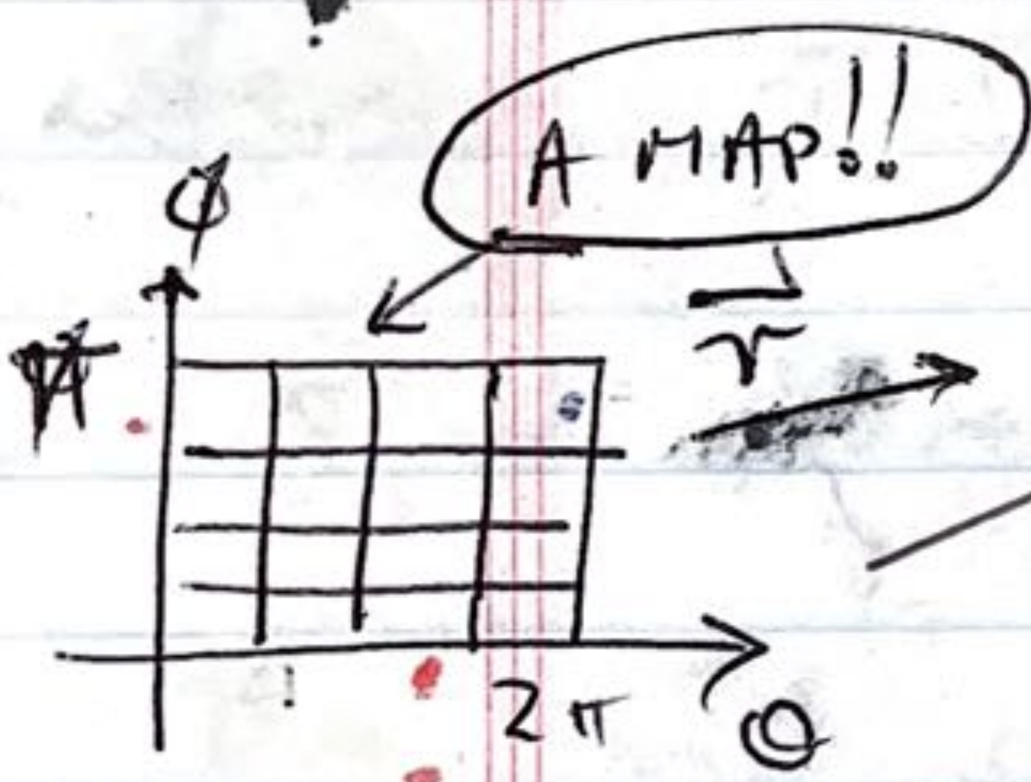
If you plug x,y,z into ⊛ eqn holds ✓ DO IT!! *

GRID CURVES

$\theta=\theta_0$
LINE OF LONGITUDE

$\phi=\phi_0$
LINE OF LATITUDE





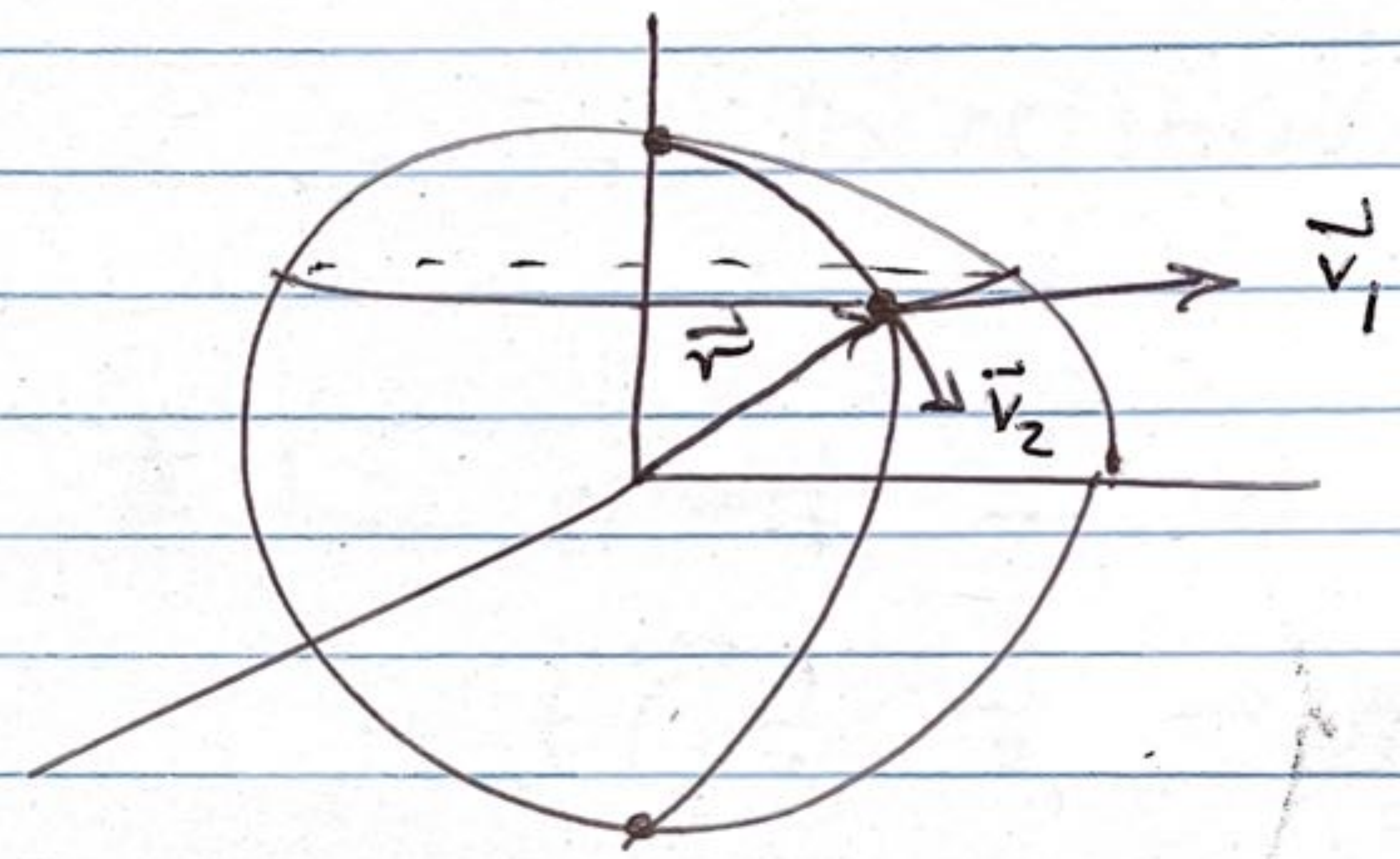
So the parameters $(u,v)=(\phi,\theta)$ give us a coordinate system on the surface.
"If you know ϕ,θ you know where you are on the sphere"

$$\vec{r}(\theta,\phi) = (R\sin\phi\cos\theta,\ R\sin\phi\sin\theta,\ R\cos\phi)$$

TANGENT PLANES TO PARAMETRIZED SURFACES

Lets parametrize the tangent plane to sphere at

$\theta = \pi/4, \quad \phi = \pi/4$



ANS

$$P(s,t) = \vec{r}(\pi/4, \pi/4) + s\vec{v}_1 + t\vec{v}_2$$

where $\vec{r}(\pi/4, \pi/4) = R\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}}\right)$.

$\vec{v}_1$ = Tangent Vector to $\phi = \pi/4$ grid curve

This grid curve has parametrization

$$\alpha(\theta) = \vec{r}(\theta, \pi/4)$$

So $\vec{v}_1 = \alpha'(\pi/4) = \frac{\partial \vec{r}}{\partial \theta}(\pi/4, \pi/4)$

$$= (-R\sin\phi\sin\theta, R\sin\phi\cos\theta, 0)\Big|_{(\theta,\phi)=\left(\frac{\pi}{4},\frac{\pi}{4}\right)}$$

$= (-1/2, 1/2, 0)$ HORIZONTAL!

AND $\vec{v}_2$ = Tangent Vector to grid curve $\theta = \pi/4$

SO $\vec{v}_2 = \frac{\partial r}{\partial \phi}(\pi/4, \pi/4) = \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{\sqrt{2}}\right)$

POINTS DOWN.

So

$$P(s,t) = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}}\right) + s\left(-\frac{1}{2}, \frac{1}{2}, 0\right)$$
$$+ t\left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{\sqrt{2}}\right)$$

parametrizes T. Plane.

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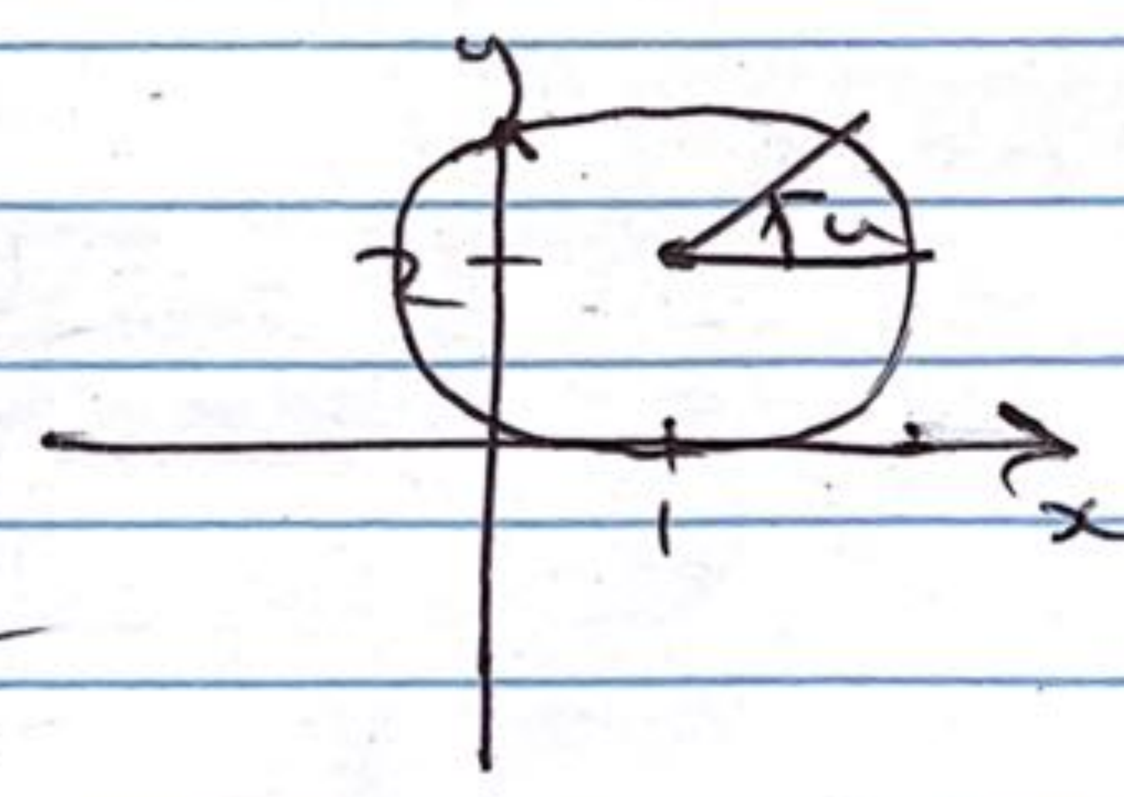
4 CYLINDER $(x-1)^2 + (y-2)^2 = 4$. ($*$)

Parameters u = Angle
v = Height = z



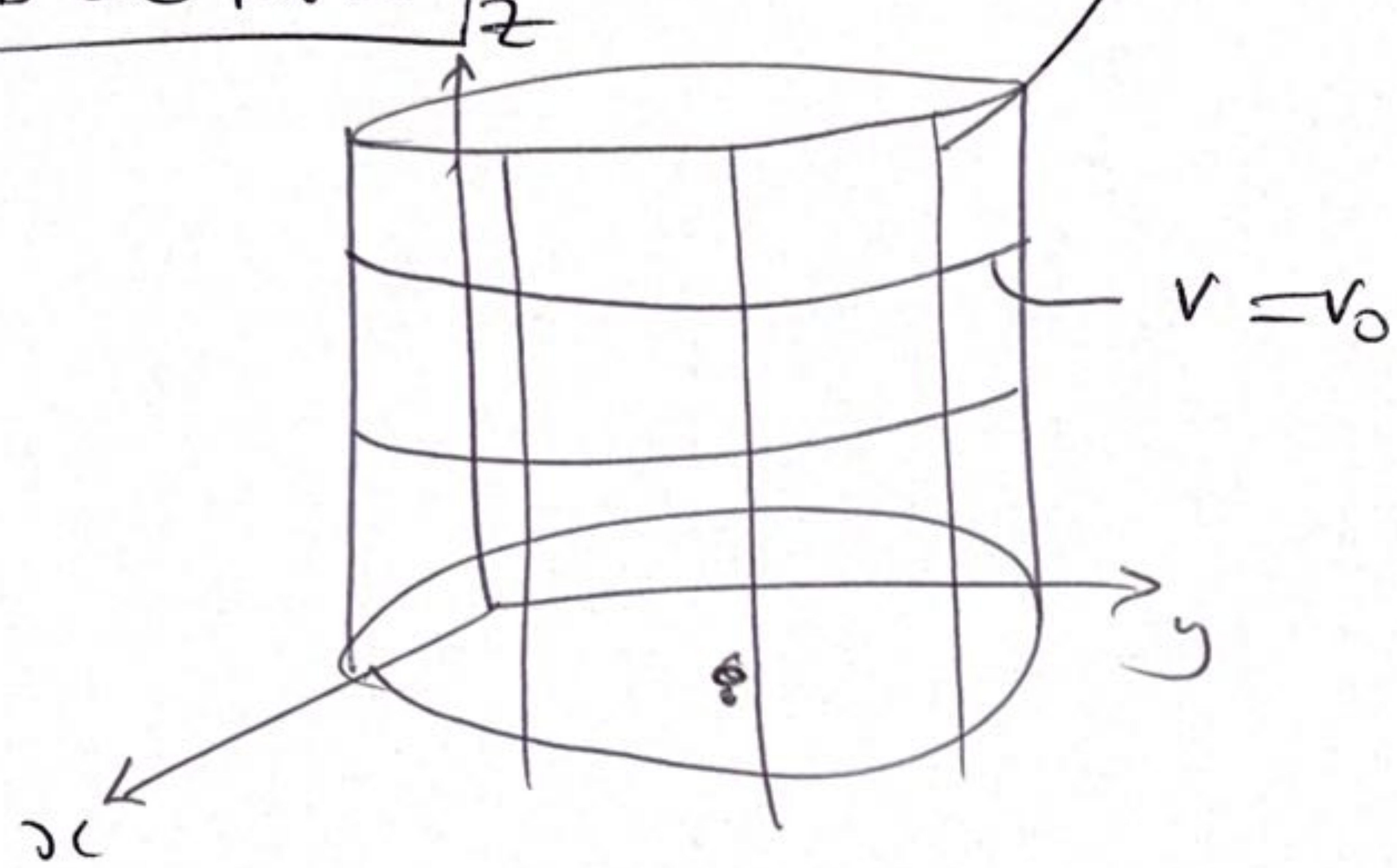
"If you know u, v you know where you are on cylinder"

From Circle:
$$\left.\begin{aligned} x-1 &= 2\cos u \\ y-2 &= 2\sin u. \end{aligned}\right\} \Rightarrow (*) \text{ HOLDS}$$

So

$$\vec{r}(u,v) = (1+2\cos u, 2+2\sin u, v) \quad \begin{aligned} 0 \le u &\le 2\pi \\ -\infty < v &< \infty. \end{aligned}$$

GRID CURVES



(5) Parametrize $z = x^2 + y^2$.

Use $(u, v) = (r, \theta)$

$x = r\cos\theta$

$y = r\sin\theta$

$z = x^2 + y^2 = r^2$

So $\boxed{\vec{x}(r, \theta) = (r\cos\theta, r\sin\theta, r^2)}$

Grid Curves are circles $(r = r_0)$ and parabolas $(\theta = \theta_0)$

(6) SHOW $x = u\cos v$, $y = u\sin v$, $z = u$ parametrizes a double cone. ELIMINATE u, v TO GET EQN IN x, y, z

$x^2 + y^2 = (u\cos v)^2 + (u\sin v)^2 = u^2 = z^2$. $\boxed{z^2 = x^2 + y^2}$