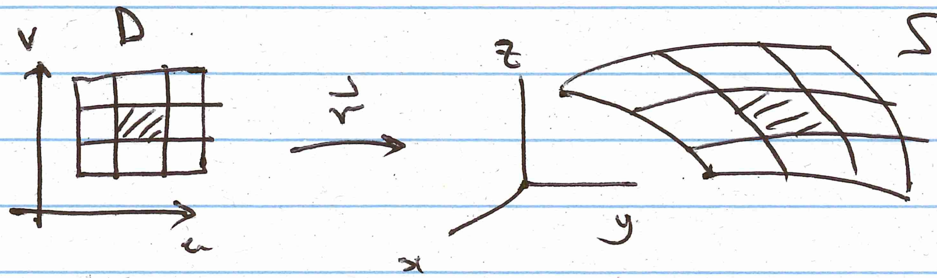
16.6, 16.7 SURFACE AREA + SURFACE INTEGRALS

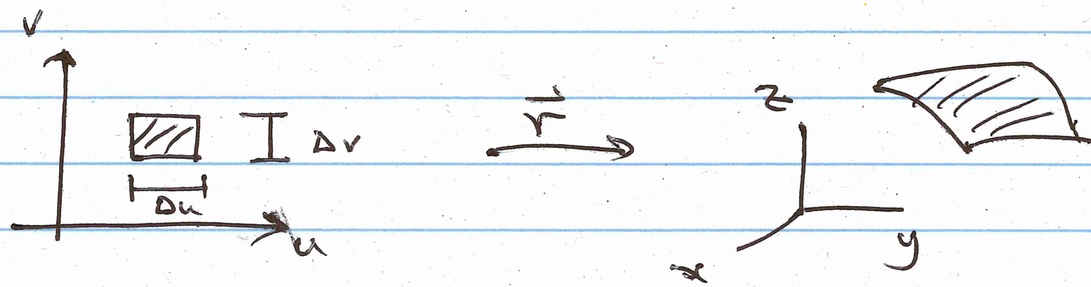
RECALL A parametrization of a surface S is

$$\vec{r} : \underset{\mathbb{R}^2}{\overset{D}{\cap}} \longrightarrow \underset{\mathbb{R}^3}{\overset{S}{\cap}}$$



$$(x, y, z) = \vec{r}(u, v)$$

AREA



$$\text{AREA} = \Delta A = \Delta u \, \Delta v \qquad \text{AREA} = \Delta S.$$

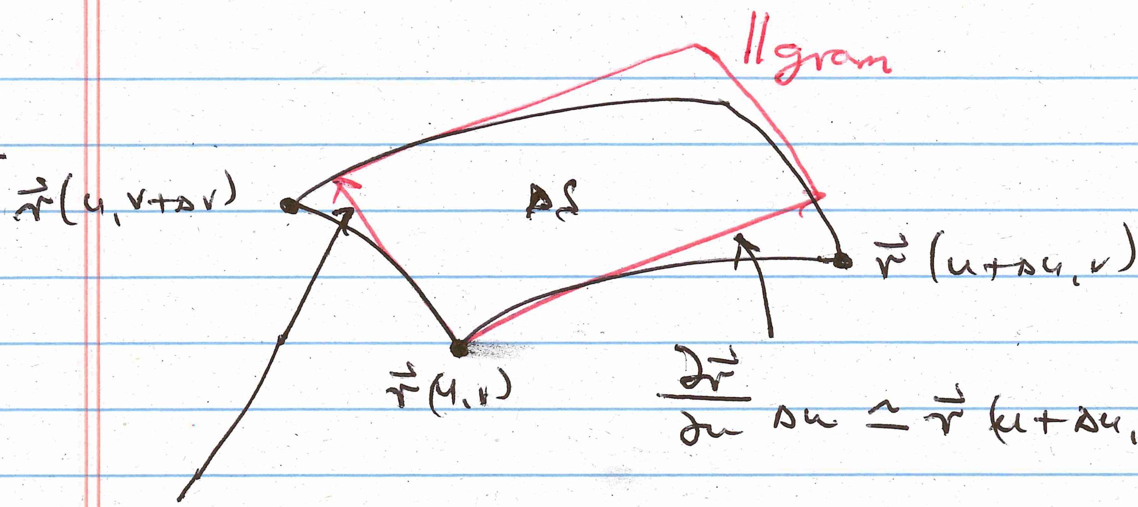
SINCE $\vec{r}$ can stretch areas, $\Delta S \neq \Delta A$.

INSTEAD

$\Delta S \simeq$ Area //gram spanned by tangent vectors $\frac{\partial \vec{r}}{\partial u} \Delta u$ and $\frac{\partial \vec{r}}{\partial v} \Delta v$ to grid curves

$$\boxed{\Delta S = \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| \Delta u \, \Delta v}$$

REASON



$\frac{\partial \vec{r}}{\partial u}\Delta u \simeq \vec{r}(u+\Delta u, v) - \vec{r}(u,v)$

$\frac{\partial \vec{r}}{\partial v}\Delta v \simeq \vec{r}(u, v+\Delta v) - \vec{r}(u,v).$

UPSHOT

1. $$\text{AREA}(S) = \iint_D \underbrace{\left|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right|}_{\text{AREA STRETCHING FACTOR}} du\,dv$$

INTEGRATION OF FUNCTIONS

2. If $w = f(x,y,z)$ is a function on $S \subset \mathbb{R}^3$
Then we define

$$\boxed{\iint_S f\,dS = \iint_D f(\vec{r}(u,v)) \left|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right| du\,dv}$$

EX If f = Density of Curved Metal ~~Pla~~ Surface S in kg/m^2

Then

$$\iint_S f\,dS = \text{Total Mass of } S.$$

ANALOGOUS TO Line Integral of function over curve C with param $(x,y,z)=\vec{r}(t)$:

$$\int_C f\,ds = \int_a^b f(\vec{r}(t))\underbrace{|\vec{r}\,'(t)|}_{\text{LENGTH STRETCHING FACTOR}}\,dt$$

SPECIAL CASE S IS GRAPH OF FUNCTION $z=g(x,y)$.

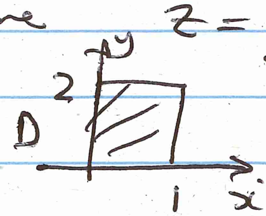
Set $\vec{r}(u,v) = (u, v, g(u,v)) = (x, y, z)$

Then
$$\frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & \frac{\partial g}{\partial u} \\ 0 & 1 & \frac{\partial g}{\partial v}\end{vmatrix}$$
$$= \left(-\frac{\partial g}{\partial u}, -\frac{\partial g}{\partial v}, 1\right)$$

So
$$\iint_S f\,dS = \iint_D f(x,y,g(x,y))\sqrt{1+\left(\frac{\partial g}{\partial x}\right)^2+\left(\frac{\partial g}{\partial y}\right)^2}\,dx\,dy$$

EXS (1) $f(x,y,z) = x^2+3z$

S is part of plane $z=g(x,y) = 1+2x+3y$ over rectangle



Then

$$\iint_S f\,dS = \int_{x=0}^{x=1}\int_{y=0}^{y=2} \left[x^2+3g(x,y)\right]\sqrt{1+\left(\frac{\partial g}{\partial x}\right)^2+\left(\frac{\partial g}{\partial y}\right)^2}\,dy\,dx$$

$$= \int_{x=0}^{1}\int_{y=0}^{2} \left[x^2+3(1+2x+3y)\right]\sqrt{1+2^2+3^2}\,dy\,dx$$

$$= \sqrt{14}\left(\frac{15}{2}+3+\frac{1}{3}\right).$$

(2) $f(x,y,z) = z^2$

S is sphere $x^2+y^2+z^2 = 1$.

Parametrize S using (θ,ϕ) of spherical coords ($\rho=1$)

So

$$(x,y,z) = \vec{r}(\theta,\phi) = (\sin\phi\cos\theta,\ \sin\phi\sin\theta,\ \cos\phi)$$

So

$$\frac{\partial\vec{r}}{\partial\phi} = (\cos\phi\cos\theta,\ \cos\phi\sin\theta,\ -\sin\phi)$$

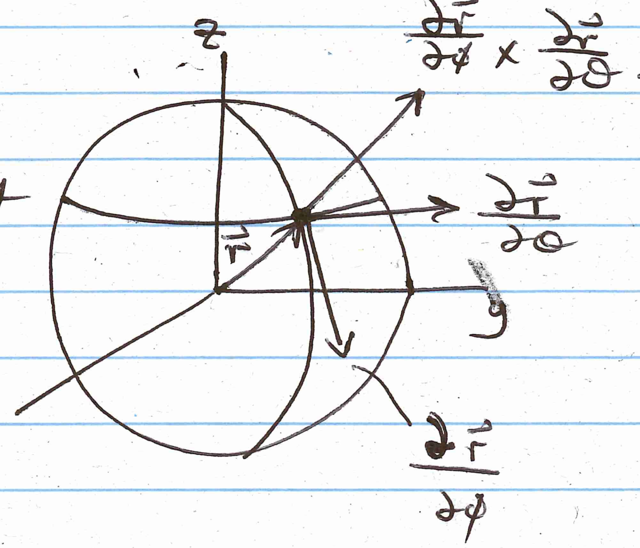
$$\frac{\partial\vec{r}}{\partial\theta} = (-\sin\phi\sin\theta,\ \sin\phi\cos\theta,\ 0)$$

VE ✓ $$\frac{\partial\vec{r}}{\partial\phi}\times\frac{\partial\vec{r}}{\partial\theta} = (\sin^2\phi\cos\theta,\ \sin^2\phi\sin\theta,\ \cos\phi\sin\phi)$$

$$= \sin\phi.\ \vec{r}(\phi,\theta)$$

GEOMETRY

On the sphere the normal vector $\frac{\partial \vec{r}}{\partial \phi} \times \frac{\partial \vec{r}}{\partial \theta}$ at a point is $\parallel$ to the position vector $\vec{r}$ of the point



UPSHOT

$$\left| \frac{\partial \vec{r}}{\partial \phi} \times \frac{\partial \vec{r}}{\partial \theta} \right| = |\sin \phi| = \sin \phi.$$

So

$$\iint_S f \, dS = \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} \cos^2\phi \cdot \sin\phi \, d\phi \, d\theta$$

$$= 2\pi \int_{-1}^{1} u^2 du \qquad (u = \cos\phi)$$

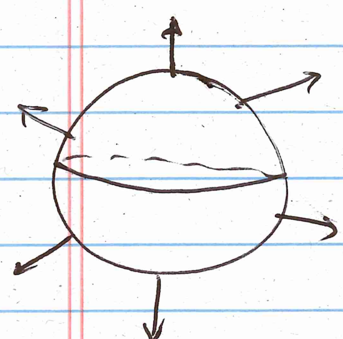
$$= 4\pi/3$$

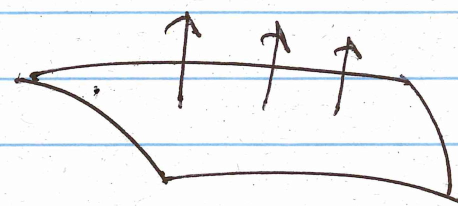
INTEGRATION OF VFs OVER SURFACES

LET S be a surface in $\mathbb{R}^3$.
Let $\vec{n}$ be a CHOICE of UNIT NORMAL VF on S.

USUALLY CHOOSE

OUTWARD NORMAL OR UPWARD NORMAL





$z = f(x,y)$

$F(x,y,z) = 0$. GRAPH OF FUNCTION.

LEVEL SURFACE

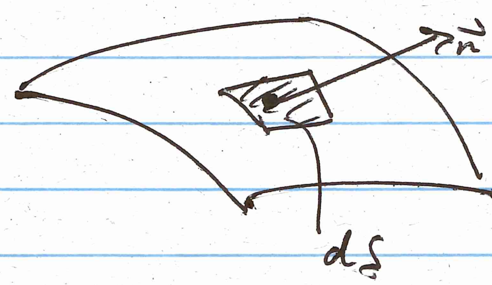
OTHER OPTIONS: IN OR DOWN

ONCE We have made a choice of $\vec{n}$ we say S is ORIENTED

DEF Let S be ORIENTED SURFACE with "positive" unit normal VF $\vec{n}$.
Let $\vec{F}$ be a VF on $\mathbb{R}^3$
Define

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S (\vec{F} \cdot \vec{n})\, dS$$

$$d\vec{S} = \vec{n}\, dS$$



$\boxed{\vec{F} \cdot \vec{n} = \text{CPT OF } \vec{F} \perp \text{ to } S}$

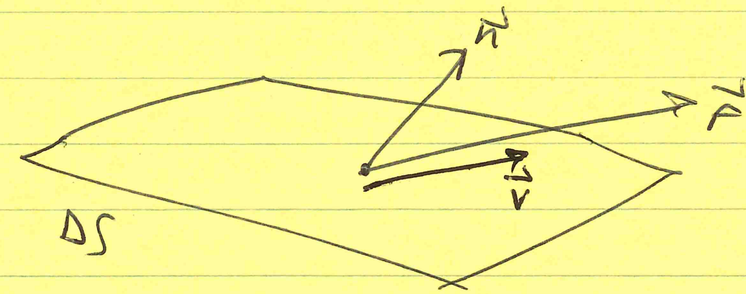
PHYSICAL MEANING

Suppose $\vec{F} = \rho \vec{v}$ where

ρ = Density = Mass / Vol of a fluid

$\vec{v}$ = Velocity VF of fluid

Let ΔS be ∥gram in $\mathbb{R}^3$ with normal $\vec{n}$.



Then

Vol of fluid crossing ΔS in dirn of $\vec{n}$ in unit time

= Volume of ∥piped given by ΔS and $\vec{v}$

= $(\vec{v} \cdot \vec{n})$ Area (ΔS)

So $(\vec{F} \cdot \vec{n})$ Area(ΔS) = Mass of fluid crossing ΔS in unit time in dirn of $\vec{n}$

For General oriented surface S:

So $\iint_S \vec{F} \cdot d\vec{S}$ = Total Mass of fluid crossing S in dirn of $\vec{n}$ in unit time

= **FLUX** of $\vec{F}$ across S in dirn $\vec{n}$.

$$\boxed{\text{HOW TO CALCULATE } \iint_S \vec{F}\cdot d\vec{S}}$$

Parametrize S by $\vec{r} : D \to \mathbb{R}^3$ so that

$$\vec{n} = + \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} \text{ is positive normal.}$$

Then

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_D \vec{F}(\vec{r}(u,v)) \cdot \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} \, |\vec{r}_u \times \vec{r}_v| \, du\, dv$$

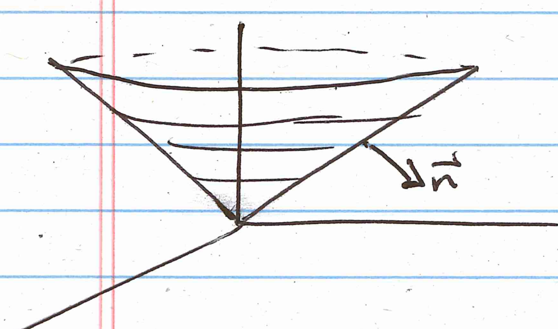
$$\boxed{\iint_S \vec{F}\cdot d\vec{S} = \iint_D \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v) \, du\, dv}$$

SCALAR TRIPLE PRODUCT

EX ① $\vec{F}(x,y,z) = x\vec{i} + y\vec{j} + z^4\vec{k}$

S is part of cone $z = \sqrt{x^2+y^2}$ below $z=1$ with DOWNWARD orientation

$\boxed{z=r}$



$\vec{r}(r,\theta) = (r\cos\theta, r\sin\theta, r)$

$0 \le \theta \le 2\pi$

$0 \le r \le 1$

$$\frac{\partial \vec{r}}{\partial r} \times \frac{\partial \vec{r}}{\partial \theta} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = (-r\cos\theta, -r\sin\theta, r)$$

UPWARD AS $r>1$.

So

$$\iint_S \vec{F}\cdot d\vec{S} = \int_{\theta=0}^{2\pi}\int_{r=0}^{1} (r\cos\theta,\ r\sin\theta,\ r^4)\ \bullet$$

$$(r\cos\theta,\ r\sin\theta,\ -r)\ dr\, d\theta$$

DOWN

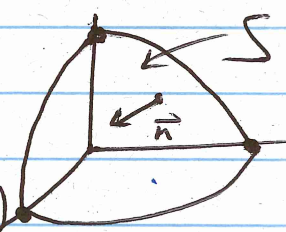
$$= \int_0^{2\pi}\int_0^1 (r^2 - r^5)\, dr\, d\theta$$

$$= \pi/3$$

(2) $\vec{F} = x\vec{i} - z\vec{j} + y\vec{k}$.

S = Part of Sphere $x^2+y^2+z^2 = 4$ in 1st octant with orientation towards origin

USE $\rho = 2$ in (ρ, ϕ, θ) formula:



$$\vec{r}(\phi, \theta) = (2\sin\phi\cos\theta,\ 2\sin\phi\sin\theta,\ 2\cos\phi)$$

$$0 \le \phi \le \pi/2$$

$$0 \le \theta \le \pi/2$$

As B4 $\quad \vec{r}_\phi \times \vec{r}_\theta = 2\sin\phi\, \vec{r} \quad$ OUTWARD.

So use $\quad \vec{r}_\theta \times \vec{r}_\phi = -2\sin\phi\, \vec{r} \quad$ INSTEAD.

Then $\displaystyle\iint_S \vec{F}\cdot d\vec{S} = \int_{\theta=0}^{\pi/2}\int_{\phi=0}^{\pi/2} \vec{F}(\vec{r}(\phi,\theta))\cdot(\vec{r}_\theta \times \vec{r}_\phi)\,d\phi\,d\theta$

$$= \int_{\theta=0}^{\pi/2}\int_{\phi=0}^{\pi/2} (2\sin\phi\cos\theta,\ -2\cos\phi,\ 2\sin\phi\sin\theta)\cdot$$
$$-2\sin\phi\,(2\sin\phi\cos\theta,\ 2\sin\phi\sin\theta,\ 2\cos\phi)$$
$$d\phi\,d\theta$$

$$= \int_{\theta=0}^{\pi/2}\int_{\phi=0}^{\pi/2} -8\sin\phi\,[\sin^2\phi\cos^2\theta - \sin\phi\cos\phi\sin\theta + \sin\phi\cos\phi\sin\theta]\,d\phi\,d\theta$$

$$= -8\left(\int_0^{\pi/2}\cos^2\theta\,d\theta\right)\left(\int_0^{\pi/2}\sin^3\phi\,d\phi\right)$$

$$= -8\left[\int_0^{\pi/2}\tfrac{1}{2}(1+\cos 2\theta)\,d\theta\right]\left[\int_0^{\pi/2}(1-\cos^2\phi)\sin\phi\,d\phi\right].$$

$$= -4\cdot\frac{\pi}{2}\cdot\left(1-\frac{1}{3}\right) = -4\pi/3.$$