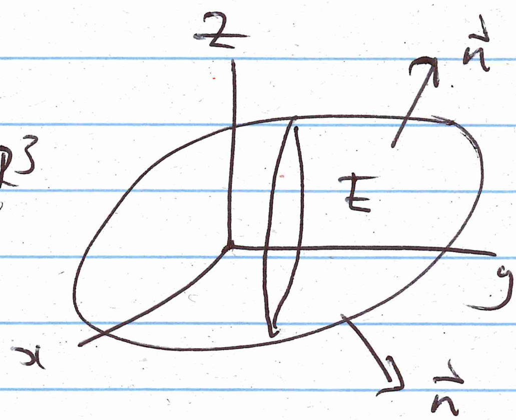
# 16.9 DIVERGENCE (GAUSS') THM

## DIV THM




Let $E$ be solid region in $\mathbb{R}^3$
Let $\partial E$ be boundary surface of $E$ with outward normal

E = WHITE MEAT OF POTATO
$\partial E$ = SKIN

Let $\vec{F}$ be VF on $\mathbb{R}^3$

Then

$$\boxed{\iiint_E (\nabla \cdot \vec{F})\, dV = \iint_{\partial E} \vec{F} \cdot d\vec{S}}$$

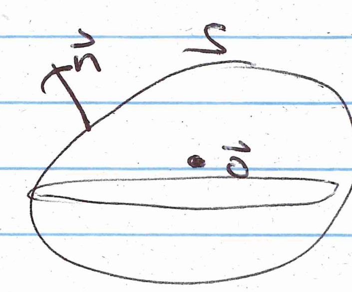
## 2 APPLICATIONS

1. Let $\vec{E} = \dfrac{\varepsilon Q \vec{r}}{|\vec{r}|^3}$ where $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ is POS'N VECTOR

be ELECTRIC FIELD due to charge $Q$ at $\vec{0}$.

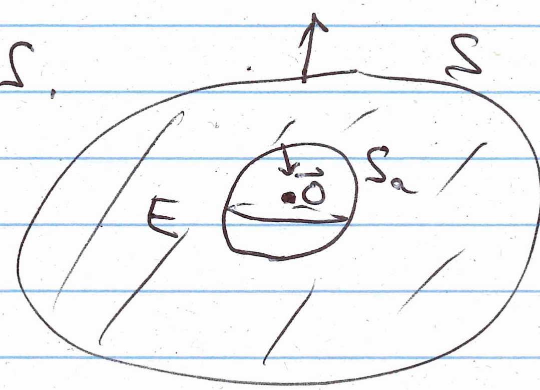
Then for ANY SURFACE $S$ that encloses the origin

$$\iint_S \vec{E} \cdot d\vec{S} = 4\pi \varepsilon Q$$



REASON Let $S_a$ = Sphere radius $a$. (small). (ORIENT OUT)

Let $E$ be solid between $S_a$ and $S$.




Then $\partial E = S - S_a$.

So by DIV THM

$$\iiint_E (\nabla\cdot\vec{E})\,dV = \iint_S \vec{E}\cdot d\vec{S} - \iint_{S_a} \vec{E}\cdot d\vec{S}$$

U⊟ ✓ $\nabla\cdot\vec{E} = 0$ on $E$.

$$\left[\ \frac{\partial}{\partial x}\frac{x}{(x^2+y^2+z^2)^{3/2}} = \frac{x^2+y^2+z^2-3x^2}{(x^2+y^2+z^2)^{5/2}} \quad \text{ETC}\ \right]$$

So $$\iint_S \vec{E}\cdot d\vec{S} = \iint_{S_a} \vec{E}\cdot d\vec{S}$$

$$= \iint_{S_a} (\vec{E}\cdot\vec{n})\,dS \qquad \vec{n} = \frac{\vec{r}}{|\vec{r}|}$$

$$= \varepsilon Q \iint_{S_a} \frac{\vec{r}}{|\vec{r}|^3}\cdot\frac{\vec{r}}{|\vec{r}|}\,dS$$

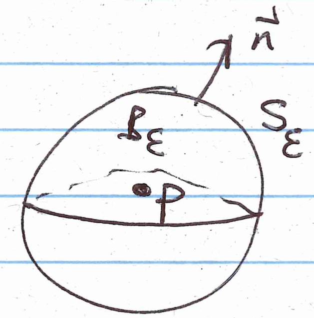
$$= \varepsilon Q \iint_{S_a} \frac{1}{|\vec{r}|^2}\,dS = \frac{\varepsilon Q}{a^2}\,4\pi a^2$$

$$= 4\pi\,\varepsilon Q.$$

2 Let $\vec{F} = \rho \vec{v}$ be MOMENTUM DENSITY VF of Fluid in $\mathbb{R}^3$.

$\rho$: DENSITY; $\vec{v}$: VELOCITY

Let $B_\varepsilon$ be ball radius $\varepsilon$ with boundary the sphere $S_\varepsilon$.




Then MASS FLUX OF $\vec{F}$ OUT OF $S_\varepsilon$

$$= \iint_{S_\varepsilon} \vec{F} \cdot d\vec{S} \overset{\text{DIV}}{\underset{\text{THM}}{=}} \iiint_{B_\varepsilon} (\nabla \cdot \vec{F})\, dV$$

$$\simeq (\nabla \cdot \vec{F})(P) \iiint_{B_\varepsilon} 1\, dV$$

Assume $\nabla \cdot \vec{F}$ is constant in $B_\varepsilon$ ($\varepsilon$ small)

$$= (\nabla \cdot \vec{F})(P)\ \mathrm{VOL}(B_\varepsilon)$$

UPSHOT

$$(\nabla \cdot \vec{F})(P) = \lim_{\varepsilon \to 0} \frac{\iint_{S_\varepsilon} \vec{F} \cdot d\vec{S}}{\mathrm{VOL}(B_\varepsilon)}$$

= FLUX DENSITY OF $\vec{F}$ AT P

= RATE AT WHICH MASS ~~LEAVES~~ EXITS P (kg/s)

$\nabla \cdot \vec{F} > 0$ at P : Fluid flows out of P

$\nabla \cdot \vec{F} < 0$ at P ______ into ______