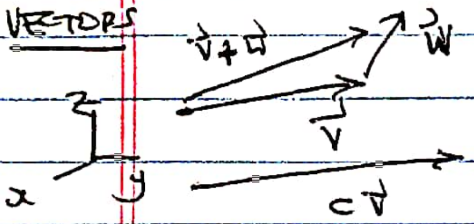
MULTIVARIABLE CALCULUS REVIEW

A VECTOR ALGEBRA

1 VECTORS



Add components of $\vec{v}$ and $\vec{w}$

Multiply components of $\vec{v}$ by c.

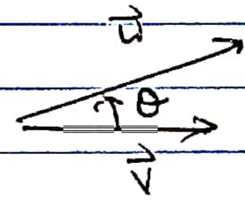
$\vec{v} = (v_1, v_2, v_3)$

$\vec{w} = (w_1, w_2, w_3)$

2 DOT PRODUCT

(a) $\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$

$= |\vec{v}|\,|\vec{w}| \cos\theta$



$|\vec{v}| = \sqrt{\vec{v}\cdot\vec{v}} = \sqrt{v_1^2 + v_2^2 + v_3^2}$

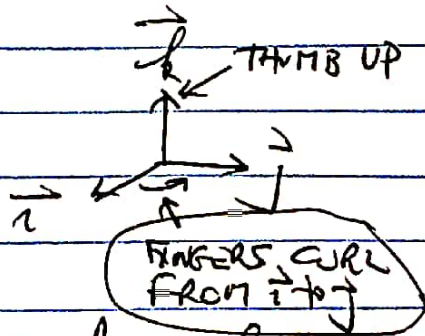
(b) $\vec{v} \cdot \vec{w} = 0 \iff \vec{v} \perp \vec{w}$ $\quad (\theta = \pi/2)$

(c) $\cos\theta = \dfrac{\vec{v}\cdot\vec{w}}{|\vec{v}|\,|\vec{w}|}$

3 CROSS PRODUCT

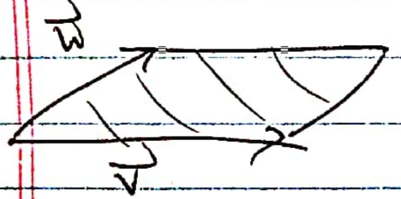
$\vec{i} \times \vec{j} = \vec{k}$

(a) $\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$

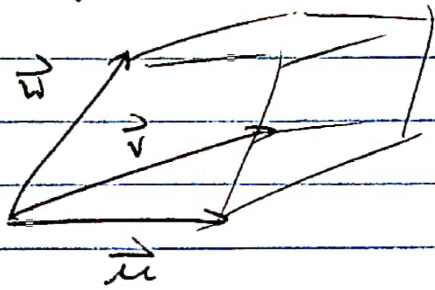


- $|\vec{v} \times \vec{w}| = |\vec{v}|\,|\vec{w}| \sin\theta$
- $\vec{v} \times \vec{w}$ is $\perp$ to both $\vec{v}, \vec{w}$, in direction of RH Rule

(b)



$$\text{AREA} = |\vec{v} \times \vec{w}|$$



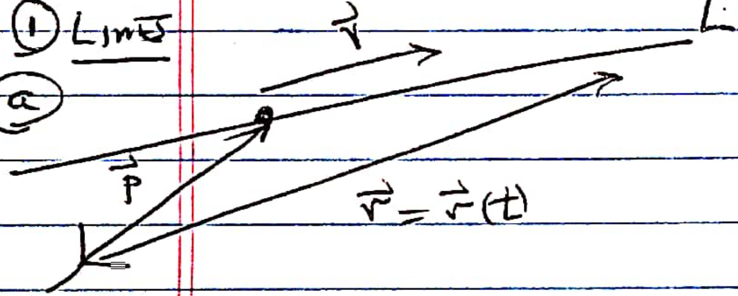
VOLUME OF SLANTED BOX

$$= |(\vec{u} \times \vec{v}) \cdot \vec{w}|$$

(B) LINES + PLANES

(1) LINES

(a)



EVERY POINT ON L IS OF FORM

$$\boxed{\vec{r} = \vec{r}(t) = \vec{p} + t\vec{v}}$$

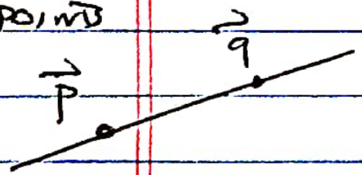
t = TIME
$\vec{r}$ = POSN AT TIME t ON L

EG $\quad p = (1, 2, 3), \quad \vec{v} = (4, 5, 6)$

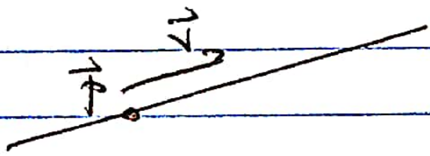
$$(x, y, z) = (1 + 4t,\ 2 + 5t,\ 3 + 6t) \qquad t \in \mathbb{R}$$

(b) WAYS TO DESCRIBE LINE

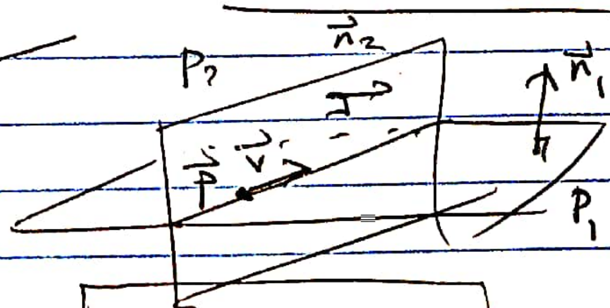
2 POINTS



POINT + VECTOR



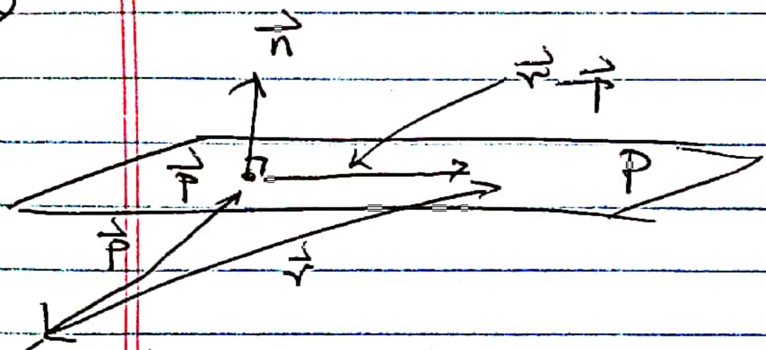
INTERSECTION OF PLANES



$$\boxed{\vec{v} = \vec{n}_1 \times \vec{n}_2}$$

(2) PLANE IF SPECIFY P USING POINT $\vec{p}$, NORMAL $\vec{n}$,

(a)



$\vec{r}-\vec{p} \perp \vec{n}$

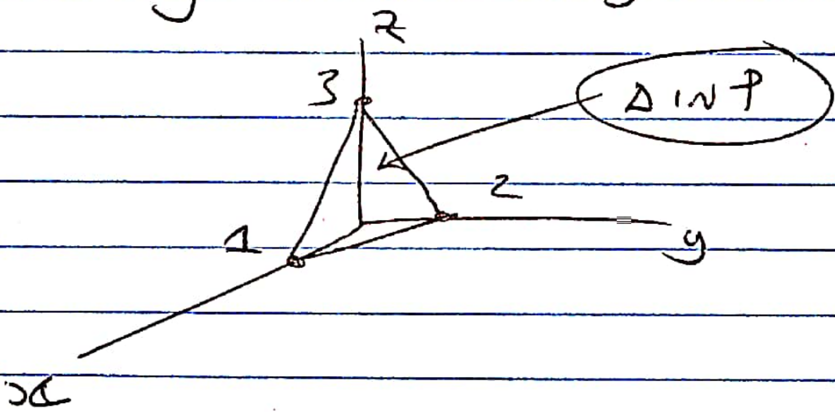
$$\boxed{(\vec{r}-\vec{p})\cdot\vec{n}=0}$$

ARBITRARY PT ON P

EX $\vec{p}=(1,2,3)$, $\vec{n}=(4,5,6)$, $\vec{r}=(x,y,z)$

(*) $\boxed{4(x-1)+5(y-2)+6(z-3)=0}$ EQN OF PLANE

(b) (i) To find point of intersection of P and z-axis set $x=0=y$ in (*) to get $z=3$.

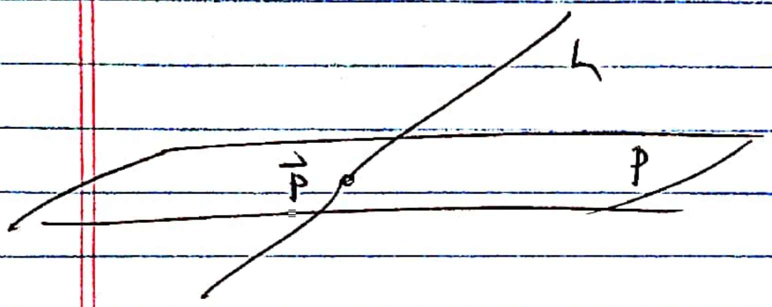


(ii) To find point on intersection of 2 planes

$$3x+4y+5z=6$$
$$x-y-z=2$$

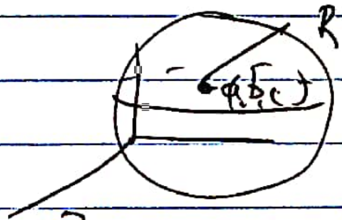
Can set $z=0$ and then solve 2 eqns in 2 unknowns, to get $(x,y,0)$ on line.

(iii) To find point of intersection of L and P



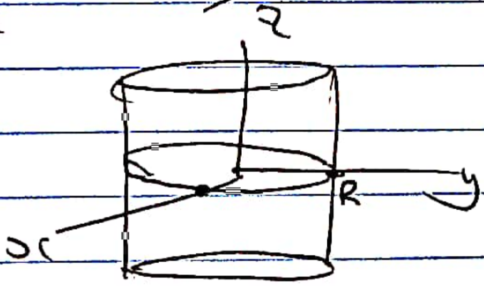
Plug parametrization $\vec{r}(t) = \vec{p} + t\vec{v}$ into equation of plane, solve for t, get $(x,y,z) = \vec{r}(t)$

⑤ QUADRIC SURFACES (Examples)



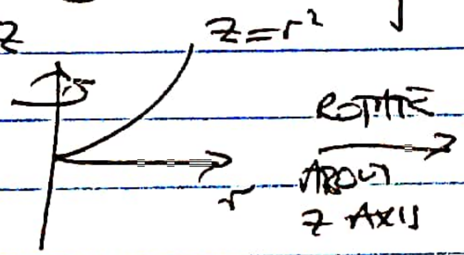
ⓐ SPHERE $(x-a)^2 + (y-b)^2 + (z-c)^2 = R^2$

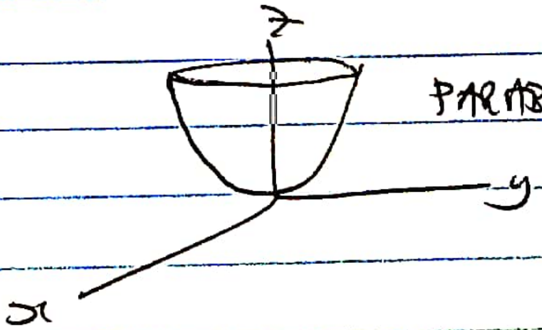
ⓑ CYLINDER ON Z-AXIS $x^2 + y^2 = R^2$



ⓒ SURFACES OF REVOLUTION

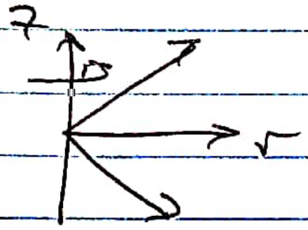
- $z = x^2 + y^2 = r^2$ in polar coords (NO θ dependence)



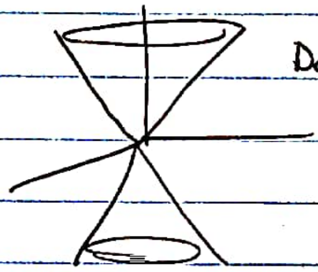


PARABOLOID

• $z^2 = x^2 + y^2 = r^2 \Rightarrow z = \pm r$

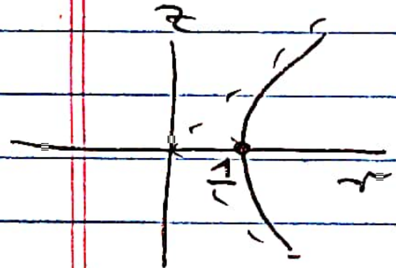


ROTATE $\longrightarrow$

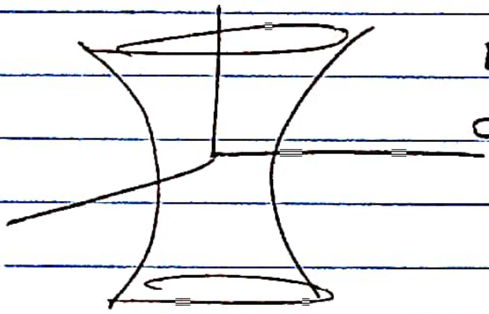


DOUBLE CONE

• $x^2 + y^2 - z^2 = 1 \Rightarrow r^2 - z^2 = 1$

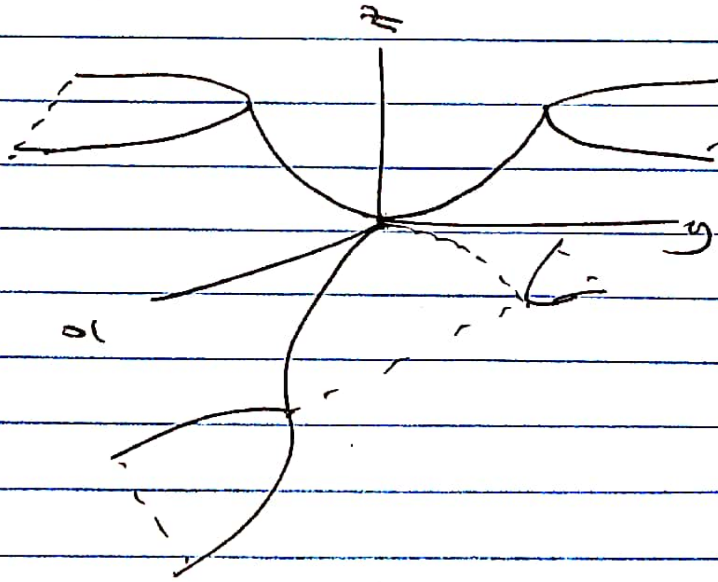


ROTATE $\longrightarrow$



HYPERBOLOID ON 1 SHEET

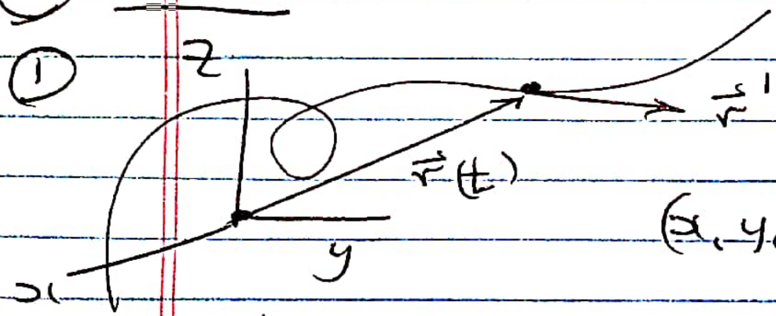
(d) SADDLE SURFACE ~~$z = x^2 - y^2$~~ $\boxed{z = y^2 - x^2}$



RIDE YOUR HORSE THIS WAY $\longleftrightarrow$

D) CURVES

1)



PARAMETRIZATION.

$(x, y, z) = \vec{r}(t)$

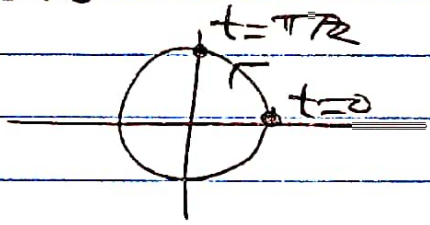
t = TIME

$\vec{r}(t)$ = POSITION AT TIME t

$\vec{r}'(t)$ = VELOCITY (VECTOR) AT TIME t

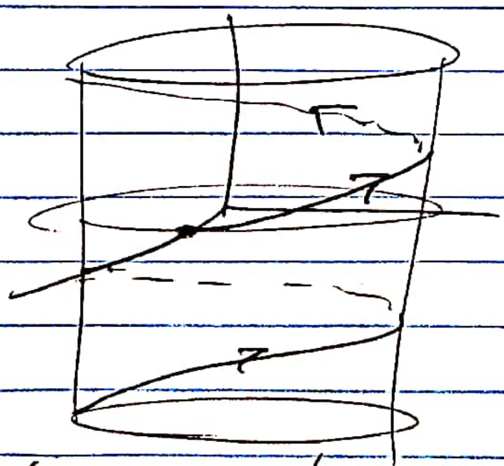
2) EXS

a) $(x, y) = \vec{r}(t) = (\cos t, \sin t)$ $0 \leq t \leq 2\pi$



CIRCLE

b) $(x, y, z) = \vec{r}(t) = (\cos t, \sin t, t)$



HELIX ON CYLINDER as

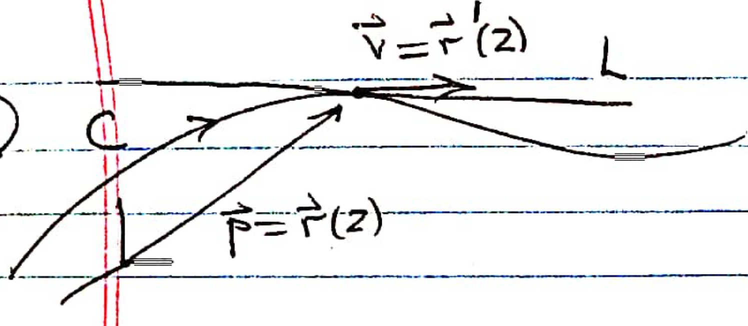
$x^2 + y^2 = \cos^2 t + \sin^2 t = 1$

$z = t$ goes up

c) $(x, y, z) = (t\cos t, t\sin t, t)$ HELIX ON CONE

as $x^2 + y^2 = t^2\cos^2 t + t^2\sin^2 t = t^2 = z^2$.

3



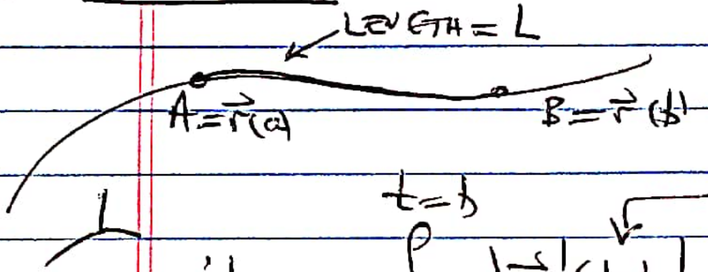
$\vec{r} = (x, y, z) = (t, t^2, t^3)$

$\vec{r}\,' = (1, 2t, 3t^2)$

Eq^n of T.L to C at $\vec{p} = \vec{r}(2)$ is

$$\vec{l}(t) = \vec{r}(2) + t\,\vec{r}\,'(2)$$
$$= (1, 4, 8) + t(1, 4, 12)$$

4 ARC LENGTH



$\boxed{\text{DIST TRAVELLED} = \text{SPEED} \times \text{TIME}}$

$$L = \int_{t=a}^{t=b} |\vec{r}\,'(t)|\,dt$$

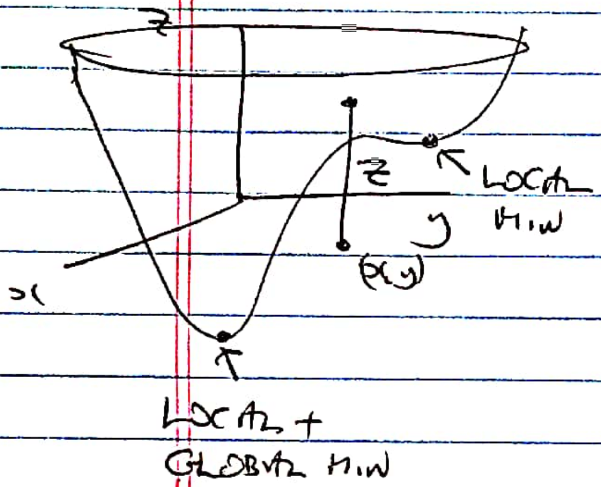
Ex $(x, y, z) = (\cos t, \sin t, t)$

$$|\vec{r}\,'(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2} = \sqrt{2}$$

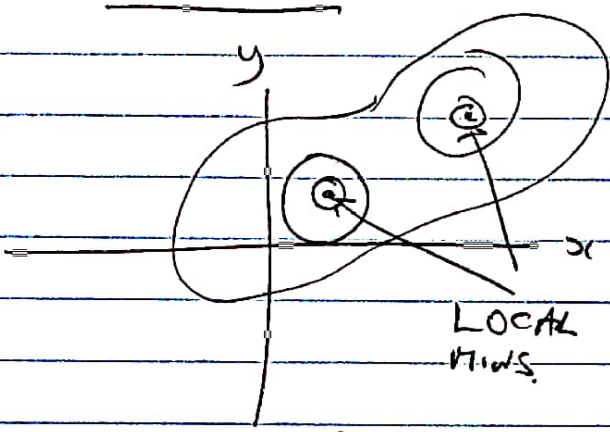
$$L = \int_a^b \sqrt{2}\,dt = \sqrt{2}(b-a)$$

E) FUNCTIONS $z = f(x, y)$

1) GRAPH OF $z = f(x,y)$



CONTOUR MAP (LEVEL CURVES)



CURVES $f(x,y) = c$

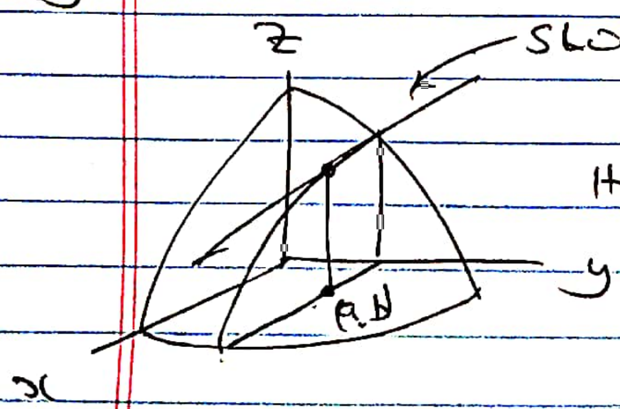
2) PARTIAL DERIVATIVES

$$z = f(x,y) = 3x^2 + 4y^2 + 5xy$$

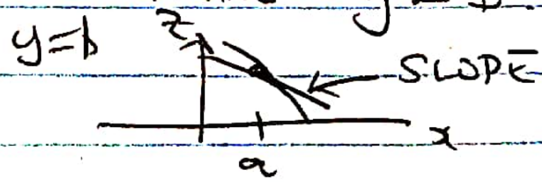
$$\frac{\partial f}{\partial x} = 6x + 5y$$

Keep y fixed. Differentiate wrt x.

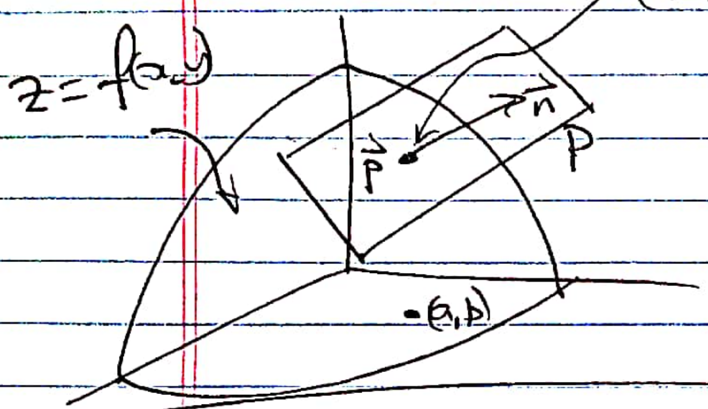
$$\frac{\partial f}{\partial y} = 8y + 5x$$



HERE: SLICING GRAPH OF $z = f(x,y)$ IN PLANE $y = b$



③ TANGENT PLANES

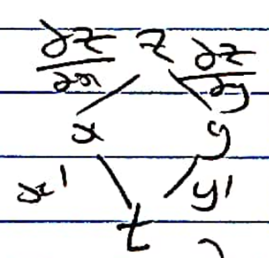


EQ OF TP TO GRAPH OF f AT $(x,y,z) = (a,b,f(a,b))$ IS

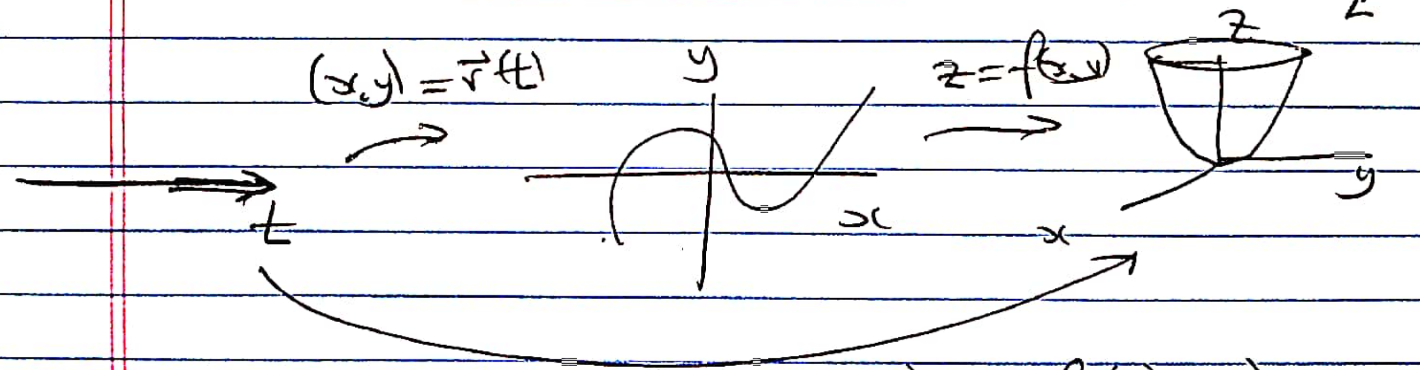
$$\boxed{z = f(a,b) + \frac{\partial f}{\partial x}(a,b)\,(x-a) + \frac{\partial f}{\partial y}(a,b)\,(y-b)}$$

ANALOGOUS TO CALC I $z = f(x)$ AT $x = a$:

$$z = f(a) + f'(a)\,(x-a)$$



④ CHAIN RULE FOR FUNCTION ON CURVE



COMPOSITION $\quad z = f(x(t), y(t)) = f(\vec{r}(t))$

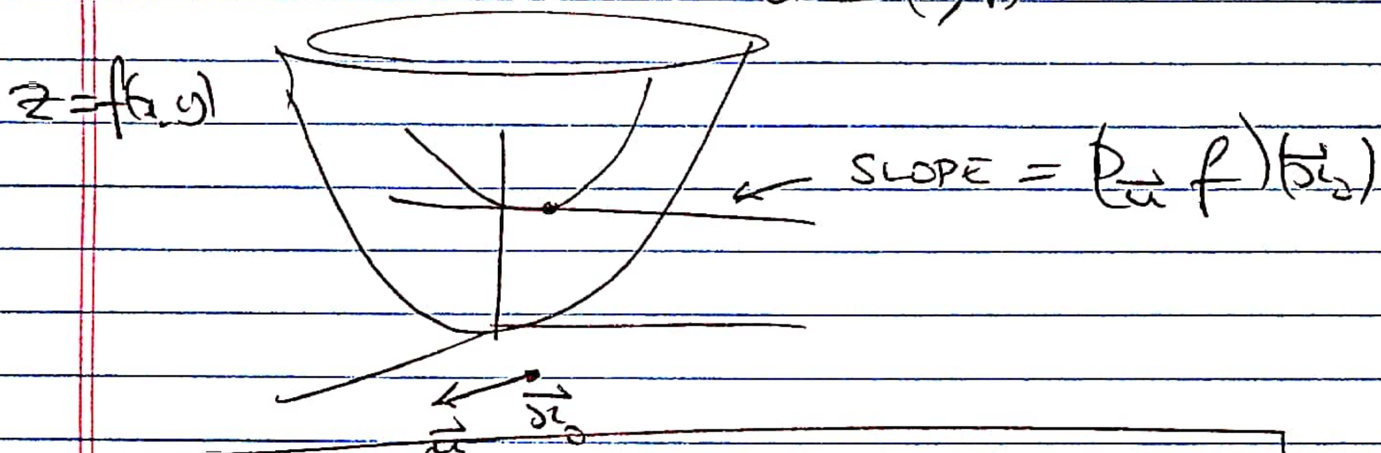
$$\frac{dz}{dt} = \frac{\partial f}{\partial x}(x(t), y(t))\,\frac{dx}{dt}(t) + \frac{\partial f}{\partial y}(x(t), y(t))\,\frac{dy}{dt}$$

OR $\quad z'(t) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$

5) GRADIENT + DIRECTIONAL DERIVATIVE FOR $z = f(x,y)$

- $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)$ GRADIENT.

- $(D_{\vec{u}} f)(\vec{x}_0)$ = RATE OF CHANGE OF f (HEIGHT) IF YOU START AT $\vec{x}_0 = (x_0, y_0)$ AND MOVE IN DIRECTION OF VECTOR $\vec{u} = (u, v)$



- $$\boxed{(D_{\vec{u}} f)(\vec{x}_0) = \nabla f(\vec{x}_0) \cdot \vec{u}}$$

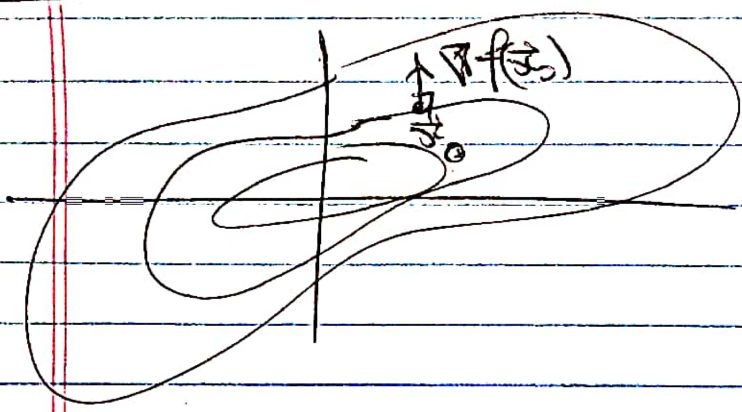
- → DIRECTION OF STEEPEST ASCENT ON GRAPH OF f

$$= \frac{\nabla f(\vec{x}_0)}{|\nabla f(\vec{x}_0)|}$$ (DIR^n VECTORS HAVE LENGTH 1)

→ RATE OF CHANGE OF f IN THAT DIR^n

$$= |\nabla f(\vec{x}_0)|$$

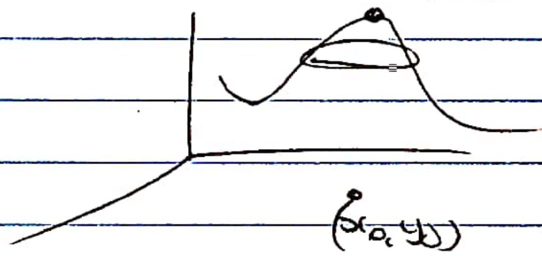
GRADIENT + LEVEL CURVES $z = f(x,y)$



$\nabla f(\vec{x}_0)$ IS A VECTOR IN (x,y) PLANE THAT IS $\perp$ TO LEVEL CURVE THRU $\vec{x}_0$.

(F) OPTIMIZATION FOR $z = f(x,y)$

(1) LOCAL OPTN



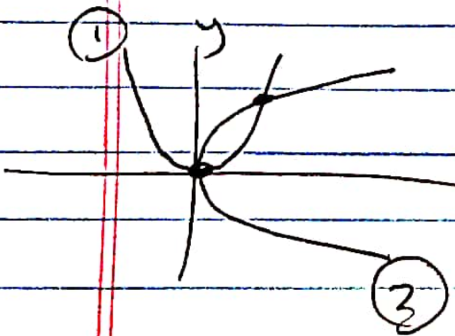
(a) CRITICAL PTS AT (x_0, y_0) SO THAT $\nabla f(x_0, y_0) = 0$.

EX $z = f(x,y) = 4 + x^3 + y^3 - 3xy$.

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \left(3x^2 - 3y,\ 3y^2 - 3x\right) = (0,0)$$

AT $\boxed{\begin{aligned} y &= x^2 \quad (1) \\ y^2 &= x \quad (2) \end{aligned}}$ 2 EQNS IN 2 UNKNOWNS

GEO



2 SOLNS $(0,0)$, $(1,1)$.

ALG PLUG (1) INTO (2) TO GET $x^4 = x$

$x(x^3 - 1) = 0 \Rightarrow x = 0, 1$

(b) 2ND DERIVATIVE TEST

SUPPOSE (x_0, y_0) IS CPT

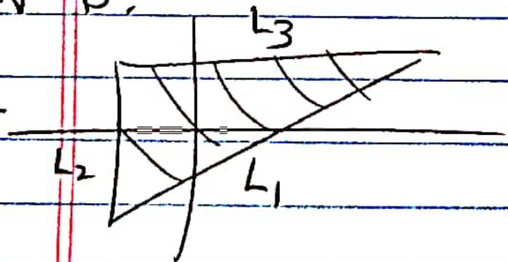
Let $D = \det \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} \Big|_{(x,y)=(x_0,y_0)}$

D	$f_{xx}(x_0,y_0)$	CLASSIFICATION	EX
+	+	LOCAL MIN	$z=x^2+y^2$
+	−	LOCAL MAX	$z=-x^2-y^2$
−	*	SADDLE POINT	$z=x^2-y^2$

(2) GLOBAL OPTIMIZATION

FIND ABS MAX / MIN OF $z = f(x,y) = x^2+2xy+3y^2$ ON D:

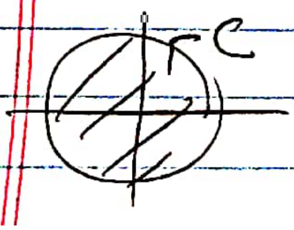
EX 1



(a) FIND CPTS OF f IN D

(b) FIND ABS MAX + MIN OF f ON L_1, L_2, L_3 (or C)

EX 2

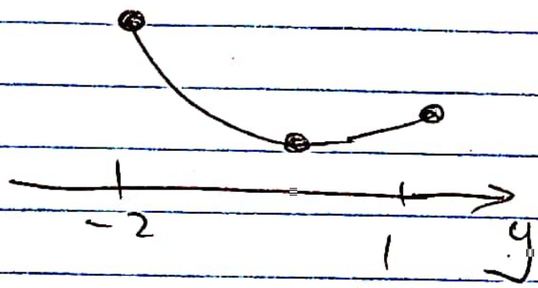


COMPARE ALL THESE VALUES.

FOR L_2 PLUG IN $x = -1$ to $z = f(x,y)$ TO GET

$$g(y) = 1 - 2y + 3y^2 \quad \text{on } [-2, 1]$$

CASE I PROBLEM



3 CPs.

FOR C PLUG $x = \cos t$, $y = \sin t$ into $z = f(x,y)$ TO GET

$$g(t) = f(\cos t, \sin t)$$

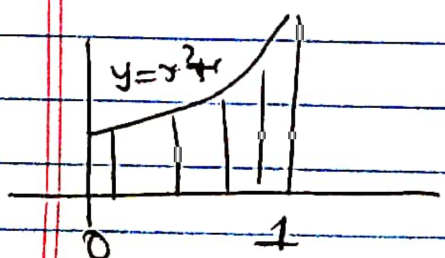
CASE I PROBLEM.

G $\boxed{\text{INTEGRALS : DOUBLE + TRIPLE}}$

1 DOUBLE

EXS $I = \iint_D x \cos y \, dA$

D BOUNDED BY $x=0$, $y=0$, $y = x^2+1$, $x=1$



$$0 \le x \le 1$$
$$0 \le y \le x^2+1$$

$$I = \int_{x=0}^{x=1} \int_{y=0}^{y=x^2+1} x \cos y \, dy \, dx$$

VERTICAL STICKS

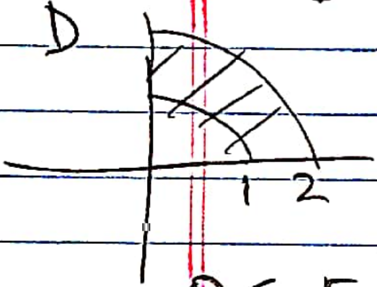
$$I = \int_{x=0}^{x=1} x \left[\sin y\right]_{y=0}^{y=x^2+1} dx$$

$$= \int_0^1 x \sin(x^2+1)\, dx = \text{ETC.}$$

3 DOUBLE IN POLAR

$$\iint_D f(x,y)\, dA = \iint_D f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

D



$$\boxed{dA = r\, dr\, d\theta}$$

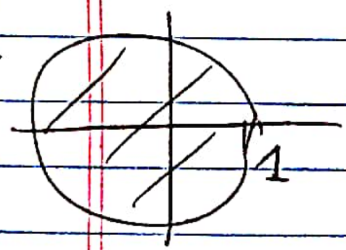
BE A PIRATE!

$0 < r < 1$

$0 < \theta < \pi/2$

EX

D



$$\iint_D e^{-x^2-y^2}\, dx\, dy$$

$$= \int_{r=0}^{1} \int_{\theta=0}^{2\pi} e^{-r^2}\, r\, dr\, d\theta$$

ETC

3 TRIPLE $dV = dx\,dy\,dz$ RECTANGULAR

$dV = r\,dr\,d\theta\,dz$ CYLINDRICAL

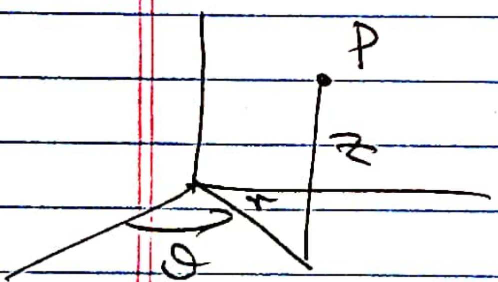
$dV = \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta$ SPHERICAL

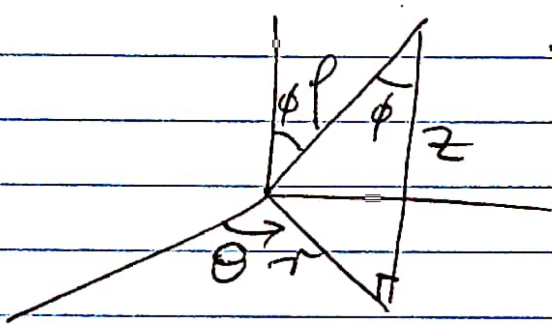
CYL COORDS

$x = r\cos\theta$
$y = r\sin\theta$
$z = z$

SPH COORDS

$x = \rho\sin\phi\cos\theta$
$y = \rho\sin\phi\sin\theta$
$z = \rho\cos\phi$



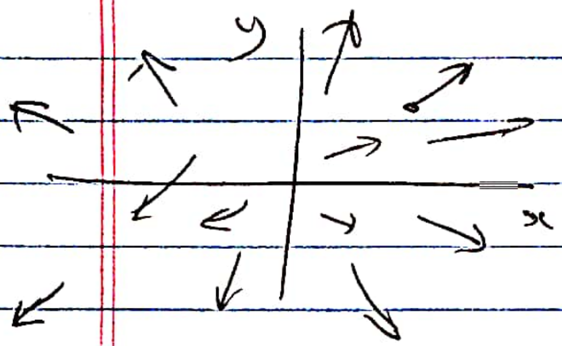


$r = \rho\sin\phi$
$z = \rho\cos\phi$

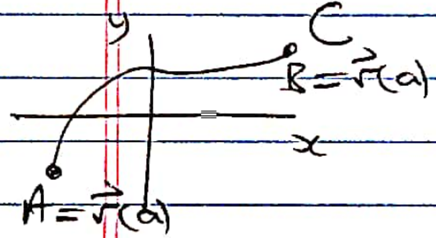
SEE EXAMPLES IN ZWECK LECTURE 21

(H) VECTOR CALCULUS

(1) VECTOR FIELDS IN $\mathbb{R}^2$: $\vec{F}(x,y) = P(x,y)\,\vec{i} + Q(x,y)\,\vec{j}$



(2) LINE INTEGRALS OF FUNCTIONS $z = f(x,y)$



$$\int_C f\,ds = \int_{t=a}^{t=b} f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt$$

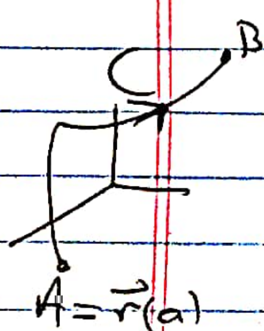
1D INTEGRAL

IF f = HEIGHT OF FENCE, C
THEN $\int_C f\,ds$ = AREA OF FENCE.

(3) LINE INTEGRALS OF VECTOR FIELDS

PATH OF PARTICLE

GIVEN
- ORIENTED CURVE C
- V.F. $\vec{F}(x,y,z)$ = FORCE ACTING ON PARTICLE



$$\boxed{\text{WORK DONE} = \int_C \vec{F} \cdot d\vec{r}}$$

WHERE $$\int_C \vec{F}\cdot d\vec{r} = \int_{t=a}^{t=b} F(\vec{r}(t))\cdot \vec{r}\,'(t)\,dt$$

CALC I INTEGRAL

EX

$$\vec{F} = x\vec{i} + 4y\vec{j}$$

$$\vec{r}(t) = t^2\vec{i} + t^3\vec{j} \qquad 0 < t < 1$$

$$\vec{F}(\vec{r}(t)) = (t^2\vec{i} + 4t^3\vec{j})$$

$$\vec{r}\,'(t) = 2t\vec{i} + 3t^2\vec{j}$$

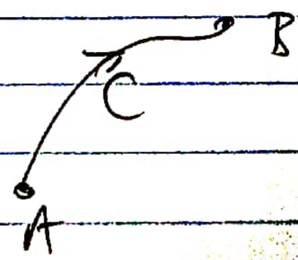
$$\text{WORK} = \int_0^1 (t^2, 4t^3)\cdot(2t, 3t^2)\,dt$$

$$= \int_0^1 2t^3 + 12t^5\,dt = \text{ETC}$$

4 FTC FOR LINE INTEGRALS

a GIVEN $z = f(x,y)$, $\vec{F} = \nabla f$ IS A VF

$$\int_C \nabla f\cdot d\vec{r} = f(B) - f(A)$$



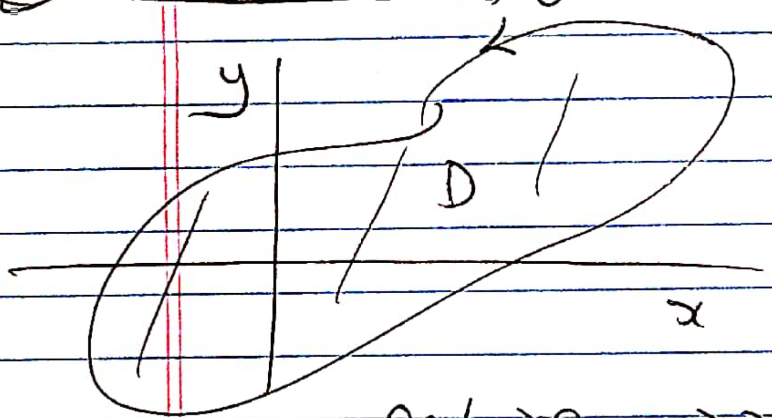
IN CONSERVATIVE VF $(\vec{F} = \nabla f)$

WORK DONE = CHANGE IN POTENTIAL ENERGY

4) $\vec{F}$ is <u>CONSERVATIVE</u> means $\vec{F} = \nabla f$

<u>THM</u> $\vec{F} = P\vec{i} + Q\vec{j}$ IS CONSERVATIVE IF AND ONLY IF $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$

5) <u>GREEN'S THM</u> $\qquad \vec{F} = P\vec{i} + Q\vec{j}$



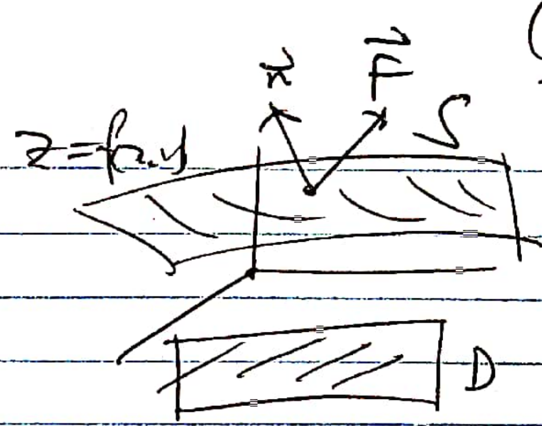
$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_{\partial D} P\,dx + Q\,dy$$

$$\qquad \qquad \qquad \| \qquad \qquad \qquad \qquad \qquad \|$$

<u>OR</u> $$\iint_D (\nabla \times \vec{F}) \cdot \vec{k}\, dA = \int_{\partial D} \vec{F} \cdot d\vec{r}$$

$$\nabla \times \vec{F} = \mathrm{CURL}(\vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix}$$

(Here $R = 0$)



6 SURFACE INTEGRALS

SURFACE $z = g(x,y)$

$\vec{F} = \vec{F}(x,y,z)$ VF ON $\mathbb{R}^3$.

FLUX OF VF OVER SURFACE

$$= \iint_S \vec{F} \cdot d\vec{S} = \iint_S (\vec{F} \cdot \vec{n})\, dS$$

= SUM OF NORMAL COMPONENT OF $\vec{F}$ OVER S

$$= \iint_D \underbrace{(P\vec{i} + Q\vec{j} + R\vec{k})}_{\vec{F}} \cdot \underbrace{\left(-\frac{\partial g}{\partial x}\vec{i} - \frac{\partial g}{\partial y}\vec{j} + \vec{k}\right)}_{``\vec{n}"} dA$$

IN CASE $\vec{n}$ POINTS UP