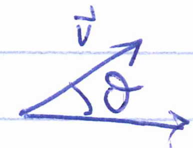
M251 LAST DAY REVIEW

A) VECTOR ALGEBRA + IT'S GEOMETRICAL MEANING

1) $\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 = |\vec{u}|\,|\vec{v}| \cos\theta$

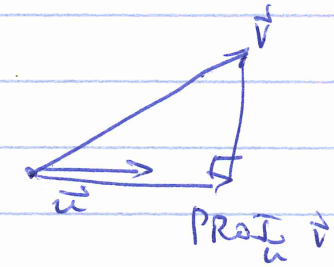


$|\vec{u}| = \sqrt{\vec{u} \cdot \vec{u}}$ = Length of $\vec{u}$

$\dfrac{\vec{u}}{|\vec{u}|}$ = UNIT VECTOR IN dirn of $\vec{u}$

$\vec{u} \cdot \vec{v} = 0 \iff \vec{u} \perp \vec{v}$

$$\text{PROJ}_{\vec{u}}\, \vec{v} = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}|} \frac{\vec{u}}{|\vec{u}|}$$

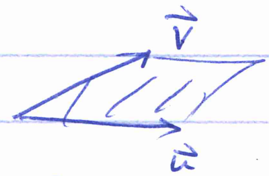


2) $$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

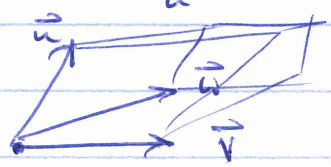
$|\vec{u} \times \vec{v}| = |\vec{u}|\,|\vec{v}| \sin\theta$

$\dfrac{\vec{u} \times \vec{v}}{|\vec{u} \times \vec{v}|}$ = Unit Vector $\perp$ to $\vec{u}, \vec{v}$, dirn given by RHRule

$|\vec{u} \times \vec{v}|$ = Area of



$|\vec{u} \cdot (\vec{v} \times \vec{w})|$ = Vol of



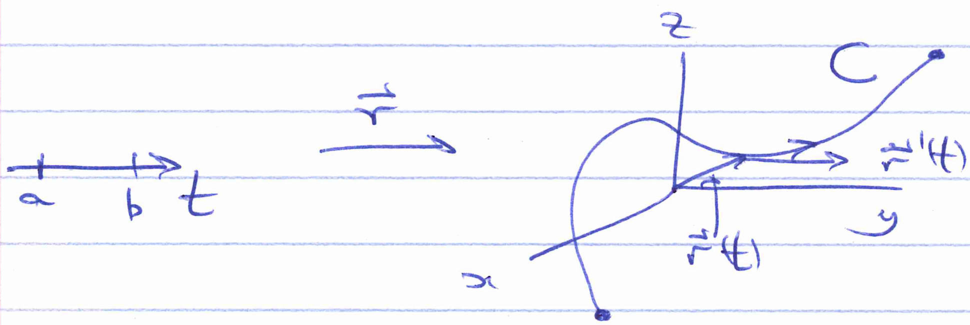
3)

B PARAMETRIZATIONS of Curves & Surfaces, S.

We use parametrizations to turn Calc problems on C, S into calculus problems on $\mathbb{R}$ and $\mathbb{R}^2$.

CURVES

$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k} \qquad \text{POSITION}$$



DERS

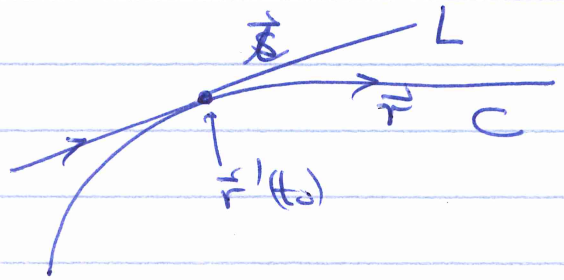
$$\vec{r}\,'(t) = \text{Velocity} = \text{Tangent Vector to } C$$

$$|\vec{r}\,'(t)| = \text{Speed}$$

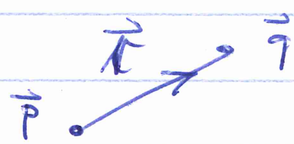
EXS 1 Fix t_0

$$\vec{L}(t) = \vec{r}(t_0) + (t-t_0)\vec{r}\,'(t_0)$$

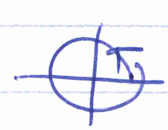
parametrizes t. line to C at $\vec{r}(t_0)$.



2 $\vec{r}(t) = \vec{p} + t(\vec{q} - \vec{p})$



3 $\vec{r}(t) = (\cos t, \sin t)$



LWB

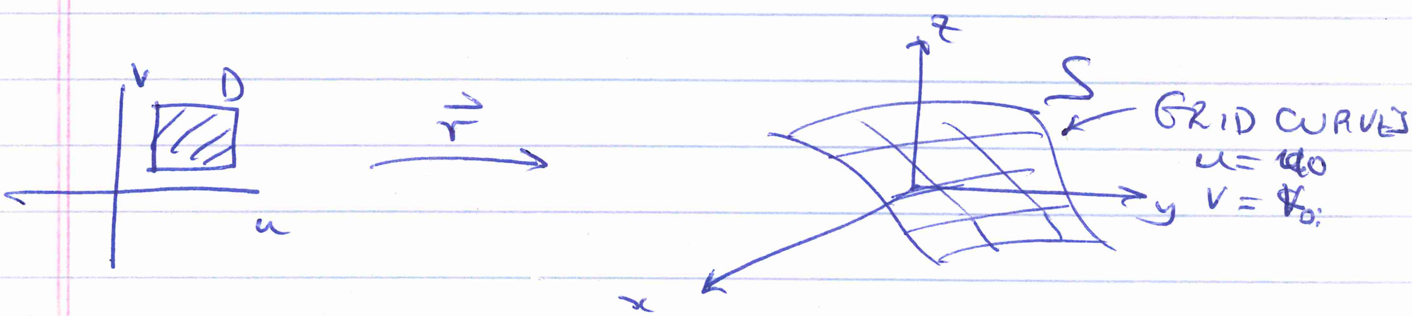
$$\int_C f\,ds = \int_a^b f(\vec{r}(t))\,\|\vec{r}\,'(t)\|\,dt$$ Used to get Mass from Mass Density.

$$\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt = \int_C \vec{F}\cdot\vec{T}\,ds$$

= CIRCULATION of $\vec{F}$ about C

SURFACES

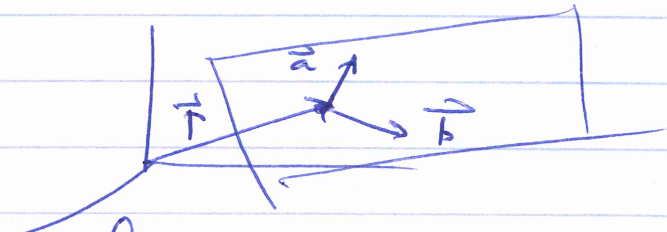
$$\vec{r}(u,v) = x(u,v)\,\vec{i} + y(u,v)\,\vec{j} + z(u,v)\,\vec{k}$$



$$S = \vec{r}\,(D).$$

EXS

① Plane $\vec{r}(u,v) = \vec{p} + u\,\vec{a} + v\,\vec{b}$

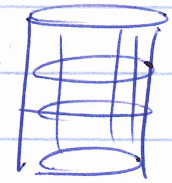


② Graph $z = f(x,y)$
$\vec{r}(u,v) = (u,\ v,\ f(u,v))$

③ Sphere $\vec{r} = \vec{r}\,(\theta, \phi)$

~~From~~ Set $\rho = 1$ in SPH COORD.

4 Cylinder $\vec{r} = \vec{r}(\theta, z)$



Set $r = 1$ in CYL coords.

ETC

DEFS

1 $\frac{\partial \vec{r}}{\partial u}(u_0,v_0)$ and $\frac{\partial \vec{r}}{\partial v}(u_0,v_0)$ are T Vectors to grid curves $v = v_0$, and $u = u_0$.

2 $\vec{P}(s,t) = \vec{r}(u_0,v_0) + s\,\frac{\partial \vec{r}}{\partial u}(u_0,v_0) + t\,\frac{\partial \vec{r}}{\partial v}(u_0,v_0)$

is param of TP to S at $\vec{r}(u_0,v_0)$.

INTS

3 $$\iint_S f\,dS = \iint_D f(\vec{r}(u,v))\overbrace{\left\|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right\|}^{\text{AS Factor.}}\,du\,dv$$

4 $$\iint_S \vec{F}\cdot d\vec{S} = \iint_D \vec{F}(\vec{r}(u,v))\cdot\left(\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right)du\,dv$$

= FLUX of $\vec{F}$ across S.

When $\vec{r}(u,v) = (x,y) = x(u,v)\,\vec{i} + y(u,v)\,\vec{j}$ 3 becomes

CofV Thm

$$\underset{\substack{\uparrow \\ \text{MORE COMPLICATED}}}{\iint_S} f(x,y)\,dx\,dy = \underset{\substack{\uparrow \\ \text{SIMPLER}}}{\iint_D} f(\vec{r}(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|du\,dv$$

(c) MAX + MIN of $z = f(x,y)$

• CHAIN RULE for

Functions on Curves $\quad (x,y) = \vec{r}(t) \quad , \quad z = f(x,y)$

$$z = (f \circ \vec{r})(t)$$

$$\frac{dz}{dt}(t) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$$

• $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$

• $\frac{\nabla f}{|\nabla f|}$ = Dirn in xy plane $f \uparrow$ most rapidly

• $|\nabla f|$ = R of C of f in This dirn.

• $\nabla f \perp$ Level Curves $f(x,y) = c$.

• Dir Der of f at $\vec{x}$ in dirn of unit vector $\vec{u}$ is

$$(D_{\vec{u}} f)(\vec{x}) = \nabla f(\vec{x}) \cdot \vec{u}.$$

• If f has MAX/MIN at $\vec{x}_0$ Then $\nabla f(\vec{x}_0) = 0$. CPt.

• 2nd Der Test tells if CPt is MAX/MIN/SADDLE/?

• LAG MULT used to MAX $z = f(x,y)$ when (x,y) constrained to be on curve C; $g(x,y) = c$.

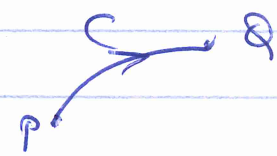
$\boxed{\text{FTC} \times 4}$

1. $\nabla \times \vec{F}$ — Def^n. — Related to <u>Circn</u>
   $\nabla \cdot \vec{F}$ — Def^n <u>Divergence</u> prop of $\vec{F}$.

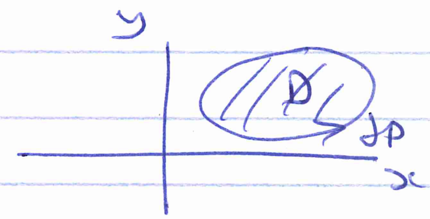
2. $\vec{F}$ cons if $\nabla \times \vec{F} = \vec{0}$.
   In this case often get $\vec{F} = \nabla f$. f = pot f^n

3. <u>FTC $\times 4$</u>

(A) $$\int_C \nabla f \cdot d\vec{r} = f(Q) - f(P)$$



(B) GT $$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_{\partial D} P\,dx + Q\,dy$$



(C) ST $$\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \int_{\partial S} \vec{F} \cdot d\vec{r}$$

(D) GT $$\iiint_E \nabla \cdot \vec{F}\, dV = \iint_S \vec{F} \cdot d\vec{S}.$$