ELECTROMAGNETISM + MAXWELL'S EQUATIONS

REF FEYNMAN'S LECTURES ON PHYSICS VOL II

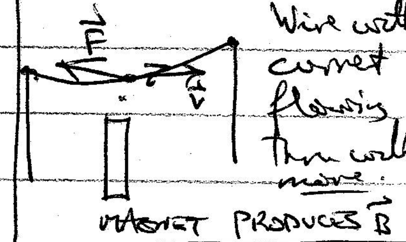
Electromagnetism describes how (moving) electric charges interact with each other

EXPERIMENTS SHOW That

(pos/neg)

The Force $\vec{F}$ on a charge q at location $\vec{x}$ that has velocity $\vec{v}$ due to its interaction with other stationary/moving charges is

$$\boxed{\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)} \quad ①$$



where $\vec{E}$, $\vec{B}$ are the Electric + Magnetic (Vector) Fields generated by the other charges ($\vec{E}$ and $\vec{B}$ depend on $\vec{x}$ due to all other charges)

NOTICE If $\vec{v} = \vec{0}$ (ELECTROSTATICS), $\vec{E} = \vec{F}/q$ = FORCE Per unit charge

IF $\vec{x}(t)$ is the position of our charged particle as a function of time t, Then Newton's 2nd Law says

$$\boxed{m \frac{d^2\vec{x}}{dt^2} = \vec{F}} \quad ② \text{ where } m \text{ is mass of particle}$$

So if we know $\vec{E}$, $\vec{B}$ we can ~~solve~~ use ① and ② to get ODE for motion of our charge

$$\boxed{m \frac{d^2\vec{x}}{dt^2} = q\left(\vec{E} + \frac{d\vec{x}}{dt} \times \vec{B}\right)}$$

Using ideas from ODEs course we can solve for $\vec{x}(t)$.

How do we find $\vec{E}$ and $\vec{B}$?

- First we need ~~to~~ equations that relate $\vec{E}, \vec{B}$ to the (other) moving charges. (MAXWELL'S EQNS)
- Then we need to convert Maxwell's equations into equations we can solve and understand (WAVE EQNS)

To understand Maxwell's eqns recall:

⓪ A closed surface S is one for which $\partial S = \emptyset$. Usually $S = \partial E$, $E \subset \mathbb{R}^3$

① The Flux of a V.F. $\vec{F}$ across a closed surface S

= Net outward flow across S

$$= \iint_S (\vec{F} \cdot \vec{n})\, dS = \iint_S \vec{F} \cdot d\vec{S}$$

② The CIRCULATION of $\vec{V}$ around a closed loop C

= Net rotational motion around C

$$= \int_C (\vec{F} \cdot \vec{T})\, ds = \int_C \vec{F} \cdot d\vec{s}$$

Maxwell's equations is a set of 4 equations. For each one there is a differential equation (DE), an equivalent integral equation (IE) and a physical interpretation (PI)

We describe an expt that motivates the PI

We state PI, Formulate as IE and then show how a version of FTC relates ~~connects~~ the DE to IE.

I GAUSS'S LAW

PI The flux of $\vec{E}$ through any closed surface S

$$= \frac{\text{Net Charge inside } S}{\varepsilon_0} \qquad \varepsilon_0 = \text{A constant}$$

IE $$\iint_S \vec{E} \cdot d\vec{S} = \frac{Q}{\varepsilon_0}$$

where Q = Net Charge in E, with $\partial E = S$

DEF Let $\rho = \rho(x,y,z)$ = Charge per unit volume
= Charge Density.

Then

$$Q = \iiint_E \rho \, dV$$ is net charge in E.

DE
The DE form of Gauss' Law is Then

$$\boxed{\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}} \quad \text{Max I}$$

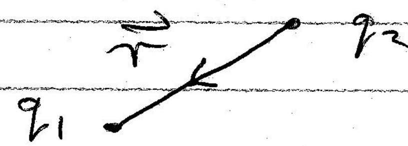
DE ⟹ IE

Suppose $\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$
Let E be a solid region with $\partial E = S$
Then

$$\iiint_E (\nabla \cdot \vec{E}) \, dV = \frac{1}{\varepsilon_0} \iiint_E \rho \, dV$$

So by Gauss Thm
$$\iint \vec{E} \cdot d\vec{S} = \frac{1}{\varepsilon_0} Q$$

Q Where does (PI) come from?
A Coulomb's Law (EXPT)



The Force $\vec{F}$ on charge q_1 due to charge q_2 is

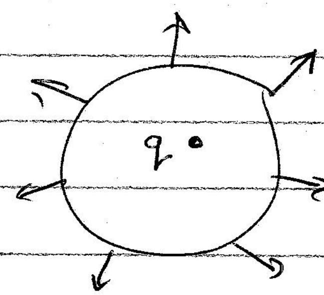
$$\vec{F} = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2 \vec{r}}{\|\vec{r}\|^3}$$

INVERSE SQUARE LAW.

where $\vec{r}$ = Vector from q_2 to q_1

So Electric Field $\vec{E}$ due to a charge $q = q_2$ is $\vec{E} = \vec{F}/q_1$

$$\boxed{\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{q\vec{r}}{\|\vec{r}\|^3}} \quad (3)$$



- Radial V.F. with length
$$\|\vec{E}\| = \frac{q}{4\pi\varepsilon_0 r^2}$$

PRINCIPLE OF SUPERPOSITION

EF due to a collection of charges is sum of EFs due to each charge.

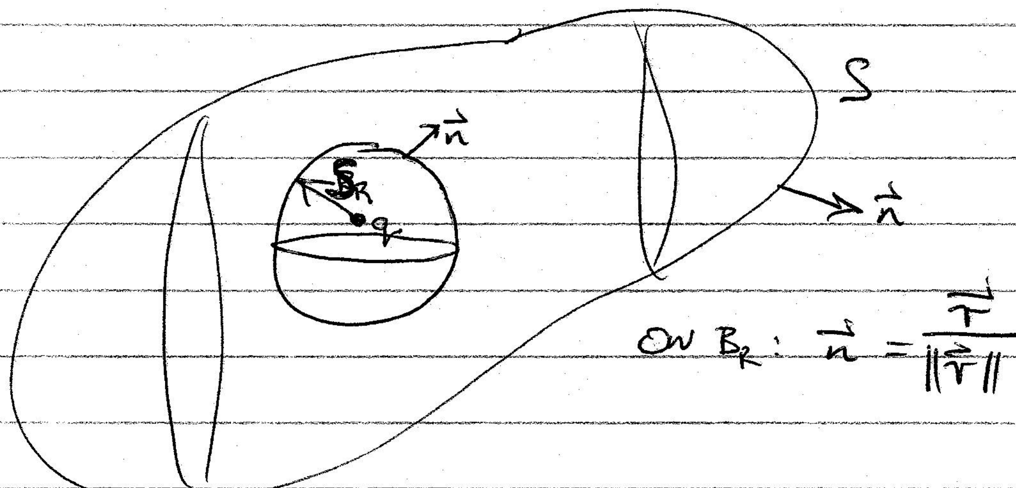
Using This (PI) follows from: Let $\vec{E}$ be EF due to single charge q. Then

$$\iint_S \vec{E}\cdot d\vec{S} = \begin{cases} 0 & q \text{ outside } S \\ \dfrac{q}{\varepsilon_0} & q \text{ inside } S \end{cases} \quad (4)$$

PROOF OF 4

A) Put q at $(0,0,0)$ inside S

Using $\vec{r} = (x,y,z)$ you can check $\nabla\cdot\vec{E} = 0$ on $\mathbb{R}^3 \setminus \{\vec{0}\}$.



Let E be solid between B_R and S, $\partial E = S - S_R$

Since $\nabla\cdot\vec{E} = 0$ on E we have

$$0 = \iiint_E \nabla\cdot\vec{E}\,dV \overset{\text{GAUSS}}{=} \iint_{\partial E} \vec{E}\cdot d\vec{S} = \iint_S \vec{E}\cdot d\vec{S} - \iint_{S_R} \vec{E}\cdot d\vec{S}$$

So

$$\iint_S \vec{E}\cdot d\vec{S} = \iint_{S_R} \vec{E}\cdot d\vec{S} = \frac{q}{4\pi\varepsilon_0}\iint_{S_R} \frac{\vec{r}}{\|\vec{r}\|^3}\cdot\frac{\vec{r}}{\|\vec{r}\|}\,dS$$

$$= \frac{q}{4\pi\varepsilon_0}\iint_{S_R} \frac{1}{\|\vec{r}\|^2}\,dS = \frac{q}{4\pi R^2\varepsilon_0}\iint_{S_R} dS$$

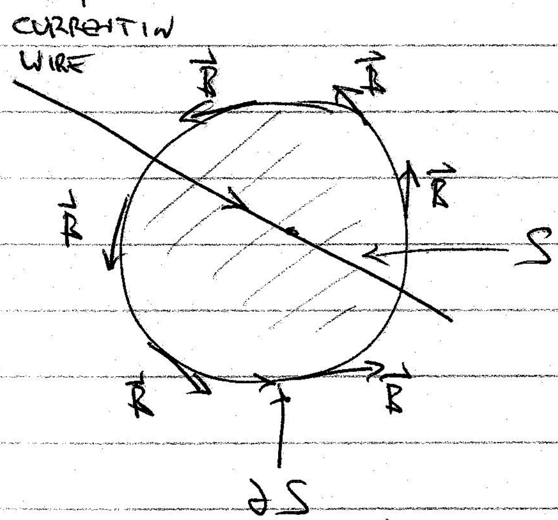
$$= \frac{q}{\varepsilon_0}.$$

B) If q outside S, $\nabla\cdot\vec{E} = 0$ on E so Gauss' Thm says

$$\iint_S \vec{E}\cdot d\vec{S} = \iiint_E \nabla\cdot E\,dV = 0.$$

II AMPERE'S LAW

A current I (ie moving charges) produces a magnetic field $\vec{B}$:



MOVE TO (*)

Let S be a surface with boundary ∂S

(PI) The Circulation of $\vec{B}$ around ∂S = $\dfrac{\text{Current through } S}{\epsilon_0 c^2}$

where c = speed of light.

LAW OF CONSERVN OF CHARGE

DEF $\vec{J}$ = Current Density

$\vec{J}/\|\vec{J}\|$ = DIRECTION of Current

$\|\vec{J}\|$ = Charge per unit area perpendicular to direction of current per unit time.

So I = Electric Current through S
= Total Charge crossing S per unit time
$= \iint_S (\vec{J}\cdot\vec{n})\,dS = \iint_S \vec{J}\cdot d\vec{S}$.

Conservation of Charge $\Rightarrow$ $I = -\dfrac{dQ}{dt}$

So $\partial E = S$: $\iint_S \vec{j} \cdot d\vec{S} = -\frac{d}{dt} \iiint_E \rho \, dV$

$$\iint_S \vec{j}\cdot d\vec{S} = \iiint_E (\nabla \cdot \vec{j})\, dV$$

So $$\boxed{\nabla \cdot \vec{j} = -\frac{\partial \rho}{\partial t}} \quad (5)$$

Div of Current Density = − RofC of Charge Density.

*

IE $$\boxed{\int_{\partial S} \vec{B}\cdot d\vec{s} = \frac{1}{\varepsilon_0 c^2} \iint_S \vec{j}\cdot d\vec{S}}$$

Since $\int_{\partial S} \vec{B}\cdot d\vec{s} \overset{\text{STOKES'}}{=} \iint_S (\nabla \times \vec{B})\cdot d\vec{S}$ ~~th~~ we get

DE $$\boxed{\nabla \times \vec{B} = \frac{\vec{J}}{\varepsilon_0 c^2}} \quad (6)$$ ~~Maxwell III~~

MAXWELL NOTICED a Problem (5) + (6)

For any VF $\vec{R}$: $\nabla \cdot (\nabla \times \vec{R}) = 0$.

So

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \vec{j} = -\varepsilon_0 c^2 \, \nabla \cdot (\nabla \times B) = 0.$$

which says CHARGES NEVER MOVE. Whoops!

Maxwell's Correction to 6 is:

$$\boxed{c^2 \nabla \times \vec{B} = \frac{\vec{J}}{\varepsilon_0} + \frac{\partial \vec{E}}{\partial t}}$$ Max II

So can produce a Mag Field $\vec{B}$ from
- Moving charges

OR
- A Time varying electric field.

III ~~No~~ Magnetic Charges Law

There is no analogue of a charge for Mag Fields.

PI The flux of $\vec{B}$ through closed $S = 0$

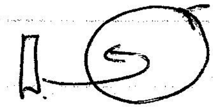
IE $$\iint_S \vec{B} \cdot d\vec{S} = 0$$

By Gauss' ~~Thm~~ This gives

DE $$\boxed{\nabla \cdot \vec{B} = 0}$$ Max III

IV Faraday's Law

Expt A time varying $\vec{B}$ produces an EF.
- If move a magnetic around a wire loop, induce a current in the wire.



(PI) Circulation of $\vec{E}$ around loop C

$= -\frac{d}{dt}$ Flux of $\vec{B}$ Through S where $\partial S = C$.

(IE) $$\int_{\partial S} \vec{E}\cdot d\vec{s} = -\frac{d}{dt}\iint_S \vec{B}\cdot d\vec{S}$$

By Stokes' Thm $$\int_{\partial S} \vec{E}\cdot d\vec{s} = \iint_S (\nabla\times\vec{E})\cdot d\vec{S}$$
~~so expect~~

(DE) $$\boxed{\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}}$$ Max IV

SUMMARY

$$\nabla\cdot\vec{E} = \rho/\varepsilon_0 \quad (1) \qquad \nabla\cdot\vec{B} = 0 \quad (3)$$
$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \quad (2) \qquad c^2\,\nabla\times\vec{B} = J/\varepsilon_0 + \partial\vec{E}/\partial t \quad (4)$$

4 PDEs for 2 unknowns $\vec{E}, \vec{B}$.
We solve them as follows:

$\nabla\cdot\vec{B} = 0 \implies \boxed{\vec{B} = \nabla\times\vec{A}}$ (5) for some $\vec{A}$.

~~Combine with $\nabla\times\vec{E}$~~

So by (2) $$\nabla\times\left(\vec{E} + \frac{\partial\vec{A}}{\partial t}\right) = \vec{0}$$

So $$\vec{E} + \frac{\partial\vec{A}}{\partial t} = -\nabla\phi$$ is conservative

or $$\boxed{\vec{E} = -\frac{\partial\vec{A}}{\partial t} - \nabla\phi} \quad (6)$$

Using 1, 4 and a clever trick we get eqns for $\phi, \vec{A}$:

NOTATION $\nabla^2\phi = \nabla \cdot \nabla\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$

WAVE EQNS

$$\boxed{\begin{aligned} \nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} &= -\frac{\rho}{\varepsilon_0} \qquad (7) \\ \nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} &= -\vec{J}/\varepsilon_0 c^2 \qquad (8) \end{aligned}}$$

NOTES

1. So if know $\rho, \vec{J}$ can solve Wave eqns to get $\phi, \vec{A}$ and hence $\vec{E}, \vec{B}$

2. 7 + 8 are examples of a PDE called The WAVE EQUATION. The solution $\phi(x,y,z,t)$ describes a wave travelling with speed c
When $\rho = 0$
One example of sol^n is $\phi(x,t) = \sin(x - ct)$

3. Maxwell computed the constant c using

$$c = \sqrt{\frac{\varepsilon_0 c^2}{\varepsilon_0}}$$

and found it equaled the speed of light!
This suggested to him that LIGHT in VACUUM ($\rho = 0, \vec{J} = \vec{0}$) is a wave ~~formed~~ consisting of coupled, time propagating varying $\vec{E}, \vec{B}$ fields!