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MATH 4362 (Spring 2018), Midterm Exam Two, (Zweck)

Instructions: This 75 minute exam is worth 75 points. No books or notes! Show all work and give **complete explanations**. Don't spend too much time on any one problem.

Throughout this exam $\chi_{[a,b]}$ is the function defined by

$$\chi_{[a,b]}(x) = \begin{cases} 1 & \text{if } a \le x \le b, \\ 0 & \text{otherwise.} \end{cases}$$

(1) [12 pts] Prove that $\{\phi_k(x) = e^{ikx} \mid k = 0, \pm 1, \pm 2, \cdots\}$ is an orthonormal set of functions on $[-\pi, \pi]$ with respect to the L^2-inner product.

$$\langle \phi_k, \phi_l \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi_k(x) \overline{\phi_l(x)}\, dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ikx} e^{-ilx}\, dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(k-l)x}\, dx$$

<u>If $k=l$</u> $\langle \phi_k, \phi_k \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} 1\, dx = \frac{1}{2\pi} \cdot 2\pi = 1$

<u>If $k \neq l$</u> $\langle \phi_k, \phi_l \rangle = \frac{1}{2\pi} \left[\frac{e^{i(k-l)x}}{i(k-l)} \right]_{-\pi}^{\pi}$

$$= \frac{1}{\pi(k-l)} \frac{e^{i(k-l)\pi} - e^{-i(k-l)\pi}}{2i}$$

$$= \frac{1}{\pi(k-l)} \sin((k-l)\pi) = 0.$$

$$\chi_{[a,b]}(x-c) = \begin{cases} 1 & a \le x-c \le b \\ 0 & \text{ELSE} \end{cases} = \begin{cases} 1 & a+c \le x \le b+c \\ 0 & \text{ELSE} \end{cases} = \chi_{[a+c,\, b+c]}(x)$$

(2) [12 pts] Find a formula for the solution $u = u(t,x)$ of the PDE initial-value problem

$$\begin{aligned} u_{tt} &= \tfrac{1}{4} u_{xx} \\ u(0,x) &= \chi_{[-1,1]}(x) = f(x) \\ u_t(0,x) &= 0. \end{aligned}$$

Sketch the solution at $t = 1$ and at $t = 4$.

$$c^2 = \frac{1}{4} \Rightarrow c = \frac{1}{2}$$

By d'Alembert's Formula

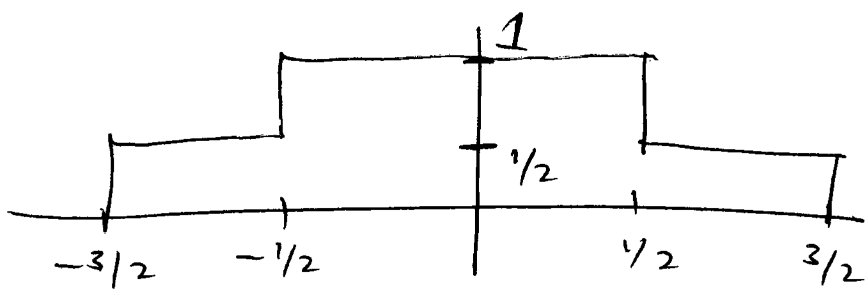
$$u(t,x) = \frac{1}{2}\left[f(x-ct) + f(x+ct) \right]$$

$$= \frac{1}{2}\left[\chi_{[-1,1]}\left(x - \frac{t}{2}\right) + \chi_{[-1,1]}\left(x + t/2\right) \right]$$

$$\overset{\text{SEE ABOVE}}{=} \frac{1}{2}\left[\chi_{[-1+t/2,\, 1+t/2]}(x) + \chi_{[-1-t/2,\, 1-t/2]}(x) \right]$$

$$u(1,x) = \frac{1}{2}\left[\chi_{[-1/2,\, 3/2]}(x) + \chi_{[-3/2,\, 1/2]}(x) \right]$$

$$= \frac{1}{2}\left(\chi_{[-3/2,\, -1/2]}(x) + 2\chi_{[-1/2,\, 1/2]}(x) + \chi_{[1/2,\, 3/2]}(x) \right)$$



PTO

(3) [12 pts] Find a formula for the solution $u = u(t, x)$ of the PDE initial-value problem

$$\begin{aligned} u_{tt} &= u_{xx} \\ u(0,x) &= 0 \quad = f(x) \\ u_t(0,x) &= \cos(x). \quad = g(x) \end{aligned} \qquad \boxed{c=1}$$

Sketch the solution at $t = \frac{\pi}{2}$ and $t = \frac{3\pi}{2}$.

By d'Alembert's Formula

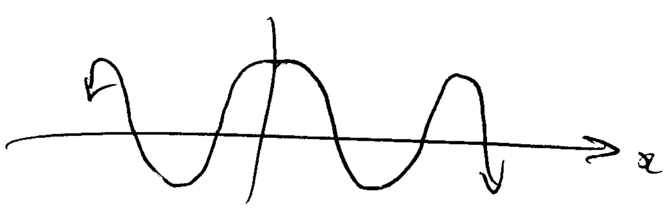
$$u(t,x) = \frac{1}{2}\int_{y=x-t}^{y=x+t} \cos(y)\,dy$$

$$= \frac{1}{2}\left[\sin y\right]_{y=x-t}^{y=x+t}$$

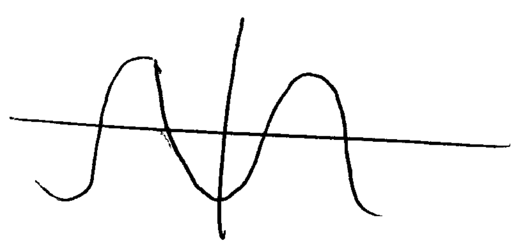
$$= \frac{1}{2}\left[\sin(x+t) - \sin(x-t)\right]$$

$$= \cos x \sin t$$ by Double Angle Formula.

$\underline{t = \pi/2}$ $\quad u(\pi/2, x) = \cos x$



$t = 3\pi/2$ $\quad u\left(\frac{3\pi}{2}, x\right) = -\cos x$



(4) [18 pts] Let $f : [-\pi, \pi] \to \mathbb{R}$ be defined by $f(x) = \chi_{[-\pi/2,\pi/2]}(x)$.

(a) Calculate the Fourier series of f.

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \chi_{[-\pi/2,\pi/2]}(x)\sin(kx)\,dx = 0$$

EVEN $\times$ ODD

$\shortparallel$

ODD

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} \chi_{[-\pi/2,\pi/2]}(x)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} 1\,dx = 1$$

$$a_k = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \cos(kx)\,dx$$

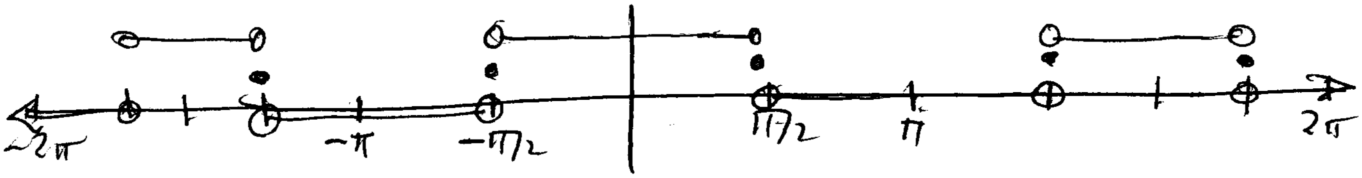
$$a_k = \frac{2}{\pi k}\sin\left(\frac{k\pi}{2}\right)$$

$$a_{2l} = 0 \quad \text{as} \quad \sin(l\pi) = 0$$

$$a_{2l+1} = \frac{2}{\pi(2l+1)}\sin\left(\frac{(2l+1)\pi}{2}\right) = \frac{2}{\pi(2l+1)}(-1)^l$$

So

$$\chi_{[-\pi/2,\pi/2]}(x) \sim \frac{1}{2} + \frac{2}{\pi}\sum_{l=0}^{\infty}\frac{(-1)^l}{2l+1}\cos[(2l+1)x]$$



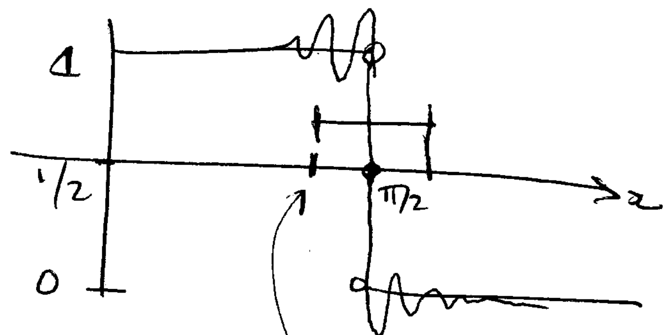
(b) Apply the theorem on pointwise convergence of Fourier series to show that the Fourier series you derived in (a) converges to a function $F : \mathbb{R} \to \mathbb{R}$. Sketch the graph of F on the domain $[-2\pi, 2\pi]$.

Thm If the 2π-periodic extension of $f : [-\pi, \pi) \to \mathbb{R}$ is piecewise C^1

Then Fourier Series of f converges to $\frac{1}{2}[\tilde{f}(x^+) + \tilde{f}(x^-)]$ where $\tilde{f}$ is the 2π-periodic extension of f.

When $f = \chi_{[-\pi/2, \pi/2]}$ $\tilde{f}$ is C^1 except at $\pm\pi/2 + 2k\pi$ where we have left (right) limits of f and f' existing. So $\tilde{f}$ is piecewise C^1.

(c) With the aid of a (rough) sketch, discuss what the Gibb's phenomenon has to say about how well the partial sums of the Fourier series of f approximate the function F near $x = \frac{\pi}{2}$.



Let S_n be nth partial sum of F.S. of f. For large enough n, S_n has an $\approx 9\%$ overshoot just to left ~~of~~ and right of $x = \pi/2$. The interval $[+\pi/2 - \delta, +\pi/2 + \delta]$ where this overshoot occurs gets narrow as n increases. Consequently $S_n \to \tilde{f}$ pointwise but not uniformly on $\mathbb{R}$.

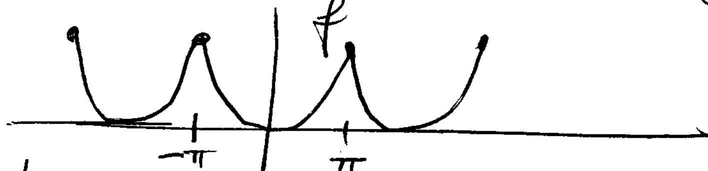
(5) [12 pts] The function $f : [-\pi, \pi] \to \mathbb{R}$ given by $f(x) = x^2$ has Fourier series

$$x^2 \sim \frac{2\pi^2}{3} + \sum_{k=1}^{\infty} \frac{4(-1)^k}{k^2} \cos(kx). \tag{1}$$

(a) Apply the theorem on the differentiation of Fourier Series to show that the Fourier series (1) can be differentiated term-by-term.

THM Says if f has a 2π-periodic extension $\hat{f}$ that is continuous and piecewise C^2 then can differentiate FS of f term by term and resulting series equals $f'(x)$. (on $[-\pi, \pi]$)

With $f(x) = x^2$



because f is EVEN,

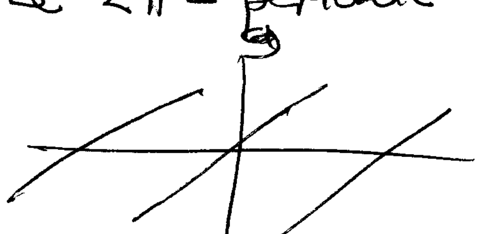
$\tilde{f}$ is CTS. clearly $\tilde{f}$ is piecewise C^2 as f is C^2

(b) What happens when you differentiate the Fourier series (1) term-by-term a second time? In particular, does the theorem on the differentiation apply to the Fourier series of the function $g(x) = f'(x) = 2x$?

We have

$$g(x) = 2x = \sum_{k=1}^{\infty} \frac{4(-1)^{k+1}}{k} \sin(kx) \quad \text{from (a)}$$

The 2π-periodic extension of g is NOT CTS



So THM above does NOT apply.

Indeed

$$2 \neq \sum_{k=1}^{\infty} 4(-1)^k \cos(kx)$$

DIVERGES AT $x = 0$

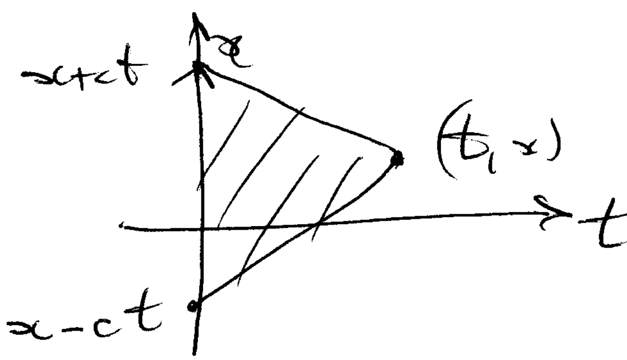
as $4(-1)^k \not\to 0$ As $k \to \infty$

(6) [9 pts] Suppose that $u = u(t, x)$ satisfies the inhomogeneous wave equation

$$\begin{aligned} u_{tt} - 9u_{xx} &= F(t,x) \\ u(0,x) &= 0 \quad = f \\ u_t(0,x) &= 0, \quad = g \end{aligned}$$

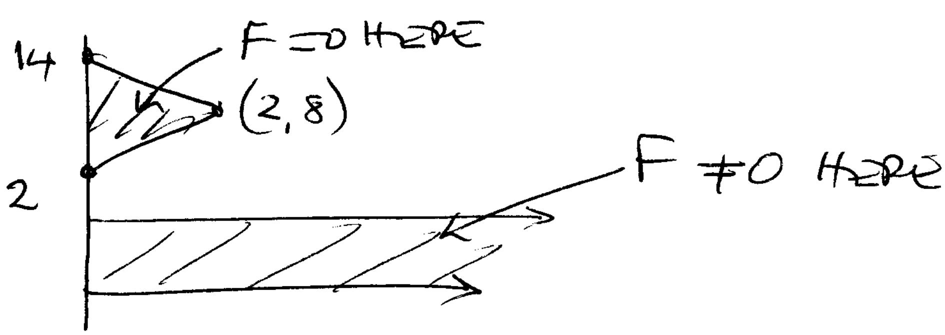
where $F(t, x) = \chi_{[0,1]}(x)$ for all $t > 0$. Use the concept of the domain of dependence to show that $u(2, 8) = 0$. Then find all x so that $u(2, x) = 0$.

$c^2 = 9 \Rightarrow c = 3$



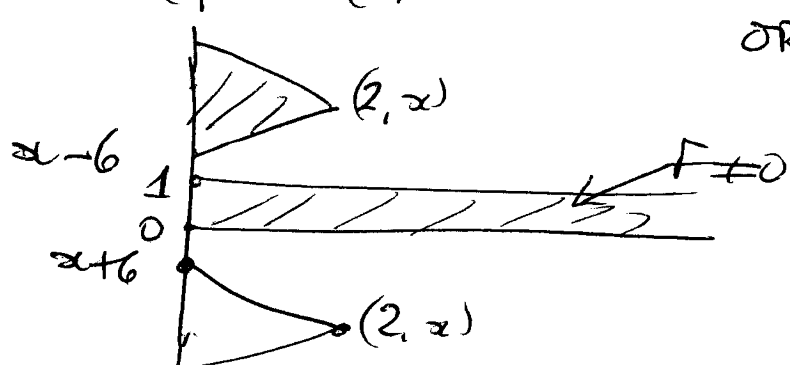
Value of u at (t, x) depends on values of f, g in $[x-ct, x+ct]$ on x axis and values of F in shaded triangle (the domain of dependence)

With $(t,x) = (2,8)$ and $c=3$ get



So $u(2,8) = 0$.

For $(t,x) = (2,x)$



Need $x - 6 > 1 \Rightarrow x > 7$

OR $x + 6 < 0 \Rightarrow x < -6$

So

$\boxed{x \in (-\infty, -6) \cup (7, \infty)}$