Math 5302, Homework 5

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1. Let $g : [a, b] \to \mathbb{R}$. Prove that $\int_a^b 1\, dg = g(b) - g(a)$.

2. Let $f : [a, b] \to \mathbb{R}$ be continuous and let $a < c < b$. Define $\widetilde{H}_c : [a, b] \to \mathbb{R}$ by

$$\widetilde{H}_c(x) = \begin{cases} 1 & \text{if } x < c, \\ 0 & \text{if } x \geq c. \end{cases}$$

   Show that $\int_a^b f(x)\, d\widetilde{H}_c(x) = -f(c)$.

3. (a) Let J be a subinterval of an interval I. Prove that if f is Riemann-Stieltjes integrable with respect to g on I then f is Riemann-Stieltjes integrable with respect to g on J. **Hint:** Use the Cauchy criterion for Riemann-Stieltjes integrability and extend tagged partitions (P, ξ), (Q, ζ) for J to tagged partitions $(\widetilde{P}, \widetilde{\xi})$, $(\widetilde{Q}, \widetilde{\zeta})$ for I.

   (b) Let $a < c < b$. Suppose that f is Riemann-Stieltjes integrable with respect to g on $[a, b]$. Then f is Riemann-Stieltjes integrable with respect to g on $[a, c]$ and on $[c, b]$ and

$$\int_a^b f\, dg \;=\; \int_a^c f\, dg \;+\; \int_c^b f\, dg.$$

4. Let $\llcorner x \lrcorner$ be the largest integer less than or equal to x. Let $b > 2$ and suppose b is not a natural number. Calculate $\int_0^b x^3\, d\llcorner x \lrcorner$.

5. Evaluate $\int_{-1}^1 f\, dg$, where

$$f(x) = \begin{cases} x^2 & \text{if } x \leq 0, \\ x^4 & \text{if } x > 0, \end{cases} \qquad \text{and} \qquad g(x) = |x|.$$

6. Evaluate $\int_{-1}^1 f\, dg$, where

$$f(x) = \begin{cases} x^2 & \text{if } x \leq 0, \\ x^4 & \text{if } x > 0, \end{cases} \qquad \text{and} \qquad g(x) = \begin{cases} x & \text{if } x \leq 0, \\ 1 - x & \text{if } x > 0. \end{cases}$$

Additional Problems (Not to be handed in)

1. Suppose that f is Riemann-Stieltjes integrable with respect to both g_1 and g_2, and let β_1, $\beta_2 \in \mathbb{R}$. Prove that f is Riemann-Stieltjes integrable with respect to $\beta_1 g_1 + \beta_2 g_2$ and that

$$\int_a^b f\, d(\beta_1 g_1 + \beta_2 g_2) \;=\; \beta_1 \int_a^b f\, dg_1 \;+\; \beta_2 \int_a^b f\, dg_2.$$

2. Suppose the f_1 and f_2 are Lipschitz continuous on $[a, b]$ and $\alpha_1, \alpha_2 \in \mathbb{R}$. Show that $\alpha_1 f_1 + \alpha_2 f_2$ is also Lipschitz continuous on $[a, b]$. How is the Lipschitz constant of $\alpha_1 f_1 + \alpha_2 f_2$ related to the Lipschitz constants of f_1 and f_2? **Definition**: The Lipschitz constant of a Lipschitz continuous function f is the least number L so that $\forall\ x_1$ and x_2 we have $|f(x_1) - f(x_2)| \leq L|x_1 - x_2|$.