Math 5302, Homework 7

John Zweck

1. Let I be a special rectangle in $\mathbb{R}^2$. Prove that the following conditions are equivalent:

   (a) $\lambda(I) = 0$,

   (b) $I^\circ = \emptyset$,

   (c) I is a subset of a line that is parallel to one of the coordinate axes.

2. Let G be open and let P be a special polygon with $P \subset G$. Prove that there exists a special polygon P' such that $P \subset P' \subset G$ and $\lambda(P) < \lambda(P')$.

3. Let G be a bounded open set in $\mathbb{R}^n$. Prove that $\lambda(G) < \infty$.

4. Use the definition of the Lebesgue measure, $\lambda(G)$, of an open set, G to prove that if

$$G \;=\; \{(x,y) \in \mathbb{R}^2 \,|\, 1 < x \text{ and } 0 < y < \frac{1}{x}\}$$

   then $\lambda(G) = \infty$.

5. Use the definition of the Lebesgue measure, $\lambda(G)$, of an open set, G to prove that if

$$G \;=\; \{(x,y) \in \mathbb{R}^2 \,|\, 0 < x \text{ and } 0 < y < e^{-x}\}$$

   then $\lambda(G) = 1$.

6. Prove that every nonempty open subset of $\mathbb{R}^2$ can be expressed as a countable union of nonoverlapping special rectangle, which may be taken to be squares:

$$G \;=\; \cup_{k=1}^{\infty} I_k,$$

   The range on k must be infinite. Why? Also prove that $\lambda(G) = \sum_{k=1}^{\infty} \lambda(I_k)$.

7. Let $\epsilon > 0$. Prove that there exists an open set $G \subset \mathbb{R}$ so that $\mathbb{Q} \subset G$ and $\lambda(G) < \epsilon$.

   [This result tells us that although G is open and contains every rational number, "most" of $\mathbb{R}$ is in G^c.]