THM 18 [R, p 130]

Let $g(x) = \sum_{n=1}^{N} A_n H_{c_n}(x)$ & $A_n \in \mathbb{R}$

where $c_n \in (a,b)$ are distinct.

Let $f : [a,b] \to \mathbb{R}$ be CTS

Then

f is RSI wrt g and

$$\int_a^b f(x)\, dg(x) = \sum_{n=1}^{N} A_n f(c_n)$$

Pf

Use Linearity and Example 17 $\square$

THM 19 Let $A_n \geq 0$ $\forall n$ and suppose $\sum_{n=1}^{\infty} A_n$ converges. Let $c_n \in (a,b)$ be distinct points

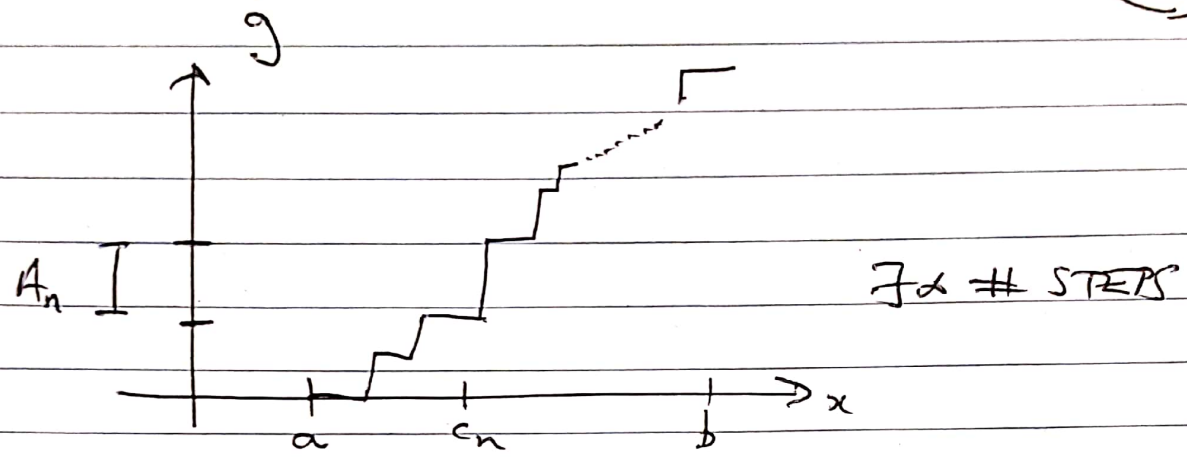
Set

$$g(x) = \sum_{n=1}^{\infty} A_n H_{c_n}(x)$$

Then g is non-decreasing and if $f : [a,b] \to \mathbb{R}$ is CTS then f is RSI wrt g and

$$\int_a^b f(x)\, dg(x) = \sum_{n=1}^{\infty} A_n f(c_n)$$

PICTURE



PF

① ⓐ $\forall x$, $g(x)$ CONVERGES by COMPARISON TEST with $\sum_{n=1}^{\infty} A_n$.

ⓑ g is NON-DECREASING $\overset{\text{THM 11}}{\Longrightarrow}$ f RSI wrt g

ⓒ $g(a) = 0$, $g(b) = \sum_{n=1}^{\infty} A_n$

② LET $\varepsilon > 0$. Choose N: $\sum_{n=N+1}^{\infty} A_n < \varepsilon$.

Set

$$g_1 = \sum_{n=1}^{N} A_n H_{c_n}$$

$$g_2 = \sum_{n=N+1}^{\infty} A_n H_{c_n}$$

$g = g_1 + g_2$, g_1, g_2 are NON-DEC^{D}.

By Thm 18

$$\int_a^b f\,dg_1 = \sum_{n=1}^{N} A_n f(c_n)$$

$\forall \varepsilon > 0 \quad \exists N:$

So $\left| \int_a^b f\,dg - \sum_{n=1}^{N} A_n f(c_n) \right|$

$= \left| \int_a^b f\,dg_2 \right|$

$\leq \max_{[a,b]} |f| \; \mathrm{VAR}_{[a,b]}(g_2)$ by Thm 5

$= (\max |f|)\,[g_2(b) - g_2(a)]$

$= (\max |f|) \sum_{N+1}^{\infty} A_n < (\max |f|)\,\varepsilon.$

So by def^n of convergence of series

$$\int f\,dg = \sum_{n=1}^{\infty} A_n f(c_n)$$

$\square$

THM 20 [WK, THM 9.27]

Let $f : [a,b] \to \mathbb{R}$ be CTS and $c \in (a,b)$.

Let $g : [a,b] \to \mathbb{R}$ be differentiable EXCEPT at $x = c$

and suppose

$$g(c^{\pm}) = \lim_{x \to c^{\pm}} g(x) \quad \exists$$

and g' is bounded on $[a,c) \cup (c,b]$.

and g' is RI on $[a,b]$

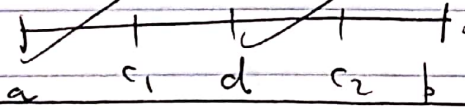
Then f is RSI wrt g and

$$\int_a^b f\,dg = \int_a^b f(x) g'(x)\,dx + f(c)[g(c^+) - g(c^-)]$$

NOTE

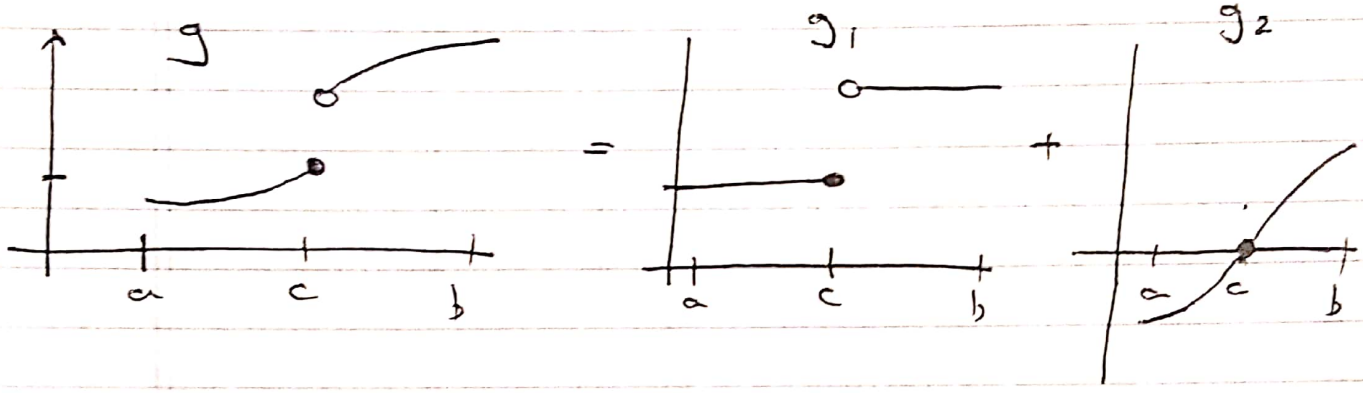
Result can be extend to case g differentiable except at $x = c_1, c_2, \ldots, c_m$ ~~using~~

~~PROP 6 (c)~~ $\cancel{\int_a^b f\,dg = \int_a^d f\,dg + \int_d^b f\,dg}$



SEE [WK] FOR DETAILS

PICTURE OF g IN CASE $g(c^-) = g(c)$



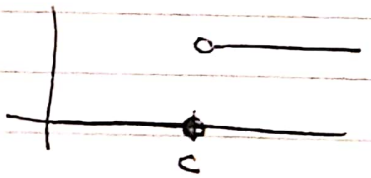
PF

IDEA Write $g = g_1 + g_2$ where g_1 is a step function (EX 17) and g_2 is CTS at $x=c$ with $g_2' = g'$ except possibly at $x = c$

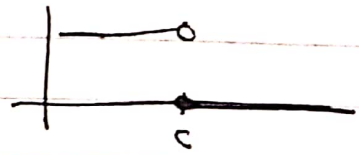
TECHNICALLY We need to be careful as $g(c) \neq g(c^-)$ in general.

Recall

$$H(x-c) = \begin{cases} 0 & x \leq c \\ 1 & x > c \end{cases}$$



$$H(c-x) = \begin{cases} 0 & x \geq c \\ 1 & x < c \end{cases}$$



Define

$$g_1(x) = [g(c^+) - g(c)]\,H(x-c) - [g(c) - g(c^-)]\,H(c-x).$$

CLAIM $g_2 = g - g_1$ is CTS at $x = c$

GET $g_2(x) = g_2(a) + \int_a^x g_2'(t)\,dt$

PF So $\int f\,dg_2 = \int f g_2'$ $\exists$ by THM 16

$= \int f g_2'$

$$\lim_{x\to c^-} g_2(x) = g(c^-) - [g(c^+) - g(c)].0 + [g(c) - g(c^-)]1$$

$$= g(c)$$

$$\lim_{x\to c^+} g_2(x) = g(c^+) - [g(c^+) - g(c)]1 + [g(c) - g(c^-)]0$$

$$= g(c)$$

So $\lim_{x\to c^-} g_2(x) = \lim_{x\to c^+} g_2(x) = g(c)$ ✓

$\square$

So

$$\int_a^b f\,dg = \int_a^b f\,dg_1 + \int_a^b f\,dg_2$$

CHECK!!

$$= [g(c^+) - g(c)]\,f(c) - [g(c) - g(c^-)][-f(c)]$$

$$+ \int_a^b f g_2'\,dx$$

$$= [g(c^+) - g(c^-)]\,f(c) + \int_a^b f g\,dx$$

IF CHANGE VALUE OF RI FN AT ONE POINT INTEGRAL SAME $\square$

NEXT we want to show Thm 16 applies to g_2:

Well Set $\varphi(x) = \begin{cases} g_2'(x) = g'(x) & \text{for } x \neq c \\ 0 & \text{for } x = c \end{cases}$

Since g' is assumed to be RI

and $\varphi = g'$ except at 1 pt

φ is RI

and $g_2(x) = g_2(a) + \int_a^x \varphi(t)\,dt$ holds

$= g_2(a) + \int_a^x g_2'(t)\,dt$

f is CTS hence RI

So by Thm 16 f is RSI wrt g_2

$$\int_a^b f\,dg_2 = \int_a^b f\,g_2'\,dx$$

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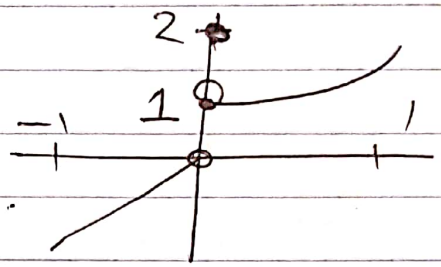
EXAMPLE 21 $[a,b] = [-1,1]$

$$g(x) = \begin{cases} x & x < 0 \\ 2 & x = 0 \\ 1+x^2 & x > 0. \end{cases}$$



$$f(x) = x^2 + 2$$

$$\int_{-1}^{1} f\,dg = \int_{-1}^{1} f g' + f(0)\left[g(0^+) - g(0^-)\right]$$

$$= \int_{-1}^{0} (x^2+2)\,1\,dx + \int_{0}^{1} (x^2+2)\,2x\,dx$$

$$+ 2\left[1 - 0\right]$$

$$= \left[\frac{x^3}{3} + 2x\right]_{-1}^{0} + \left[\frac{x^4}{2} + 2x^2\right]_{0}^{1} + 2$$

$$= \frac{1}{3} + 2 + \frac{1}{2} + 2 + 2$$

$$= 6\,5/6$$

THM 22 INTEGRATION BY PARTS

Suppose g is RSI wrt f

Then f is RSI wrt g and

$$\int_a^b f\,dg = f(b)g(b) - f(a)g(a) - \int_a^b g\,df$$

NOTES

① CASE $g = 1$, f differentiable gives

$$0 = \int_a^b f\,d1 = f(b) - f(a) - \int_a^b df$$

OR $\int_a^b f'(x)\,dx = f(b) - f(a)$ $\boxed{\text{FTC}}$

② CASE f, g DIFF'BLE gives

$$\int_a^b f g'\,dx = [fg]_a^b - \int f' g\,dx$$

- Proof of Thm 22 $\therefore$ gives proof of "Parts" that does not rely on Product Rule.

PF Let $P = \{x_k\}_{k=0}^n$, $\xi = \{\xi_k\}_{k=1}^n$

$$a = x_0 \le \xi_1 \le x_1 \le \xi_2 \le x_2 \le \dots \le x_{n-1} \le \xi_n \le x_n = b$$

AS USUAL $\xi_k \in [x_{k-1}, x_k]$

BUT ALSO $x_k \in [\xi_k, \xi_{k+1}]$.

SET $\xi_0 = a, \quad \xi_{n+1} = b$

$\tilde{x}_{k+1} = x_k \in [\xi_k, \xi_{k+1}]$

So $\tilde{x}_k \in [\xi_{k-1}, \xi_k]$.

$$a = \xi_0 = \tilde{x}_1 \le \xi_1 \le \tilde{x}_2 \le \xi_2 \le \dots \le \tilde{x}_n \le \xi_n \le \tilde{x}_{n+1} = \xi_{n+1} = b$$

We now consider $Q = \{\xi_k\}_{k=0}^{n+1}$ with tags

$\tilde{x} = \{\tilde{x}_k\}_{k=1}^{n+1}$.

We compute $\sigma(f, P, \xi, g)$ and relate it

to $\sigma(g, Q, \tilde{x}, f) = \sum_{k=1}^{n+1} g(\tilde{x}_k)[f(\xi_k) - f(\xi_{k-1})]$

<u>Well</u>

$$\sigma(f, P, \xi, g) = \sum_{k=1}^{n} f(\xi_k)[g(x_k) - g(x_{k-1})]$$

$$= -\left[\sum_{k=1}^{n} g(x_{k-1}) f(\xi_k) - \sum_{k=1}^{n} g(x_k) f(\xi_k)\right]$$

$$\cancel{= -\left[\sum_{k=0}^{n-1} g(x_k) f(\xi_{k+1}) - \sum_{k=1}^{n} g(x_k) f(\xi_k)\right]}$$

$$= -\left[\sum_{k=1}^{n} g(\hat{x}_k) f(\xi_k) - \sum_{k=1}^{n} g(\hat{x}_{k+1}) f(\xi_k) \right]$$

$$= -\left[\sum_{k=1}^{n} g(\hat{x}_k) f(\xi_k) - \sum_{k=2}^{n+1} g(\hat{x}_k) f(\xi_{k-1}) \right]$$

$$= -\left[\sum_{k=1}^{n+1} g(\hat{x}_k) f(\xi_k) - \sum_{k=1}^{n+1} g(\hat{x}_k) f(\xi_{k-1}) \right]$$

$$+ g(\hat{x}_{n+1}) f(\xi_{n+1}) - g(\hat{x}_1) f(\xi_0)$$

$$= \sigma(g, Q, \hat{x}, f) + g(b) f(a) - g(a) f(a) \quad (1)$$

NOW LET $\varepsilon > 0$

Suppose g is RSI wrt f.

Then $\exists \delta > 0 : \forall (Q, \hat{x})$ with $\|Q\| < \delta$ we have

$$\left| \sigma(g, Q, \hat{x}, f) - \int_a^b g \, df \right| < \varepsilon \quad (2)$$

Let (P, ξ) be any part^n with $\|P\| < \frac{\delta}{2}$.

Let $(Q, \hat{x})$ be constructed from (P, ξ) as above

Then $\bigcup$ ✓ $\|Q\| < \delta$ holds, ~~where Q~~

So $\left| \sigma(f, P, \xi, g) - [f(b)g(b) - f(a)g(a) - \int_a^b g\,df] \right|$

$= \left| \sigma(g, Q, \tilde{x}, f) - \int_a^b g\,df \right| < \varepsilon$ by ①, ②

So f is RSI wrt g and

$$\int_a^b f\,dg = f(b)g(b) - f(a)g(a) - \int_a^b g\,df$$

D

<u>THM 23</u> [MEAN VALUE THM]

Let $f : [a,b] \to \mathbb{R}$ be CTS and $g : [a,b] \to \mathbb{R}$ be bounded and non-decreasing. Then $\exists\, c \in [a,b]$:

$$\int_a^b f\,dg = f(c)[g(b) - g(a)]$$

<u>PF</u>

$A = \min_{[a,b]}(f)\,[g(b) - g(a)] \leq \sum_{k=1}^{n} f(\xi_k)[g(x_{k+1}) - g(x_k)]$

$\leq \max_{[a,b]}(f)\,[g(b) - g(a)] = B$

holds as f CTS so bounded, and g non-dec.

~~Since~~

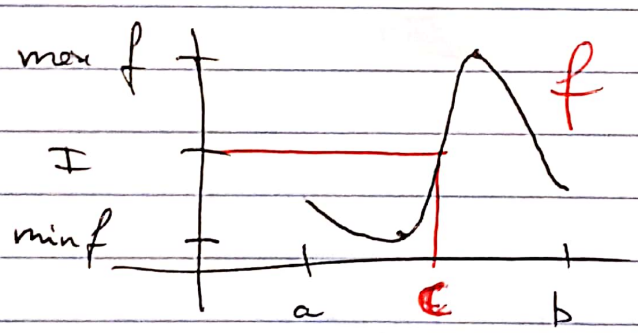
So $\forall (P, \xi)$

$$\sigma(f, P, \xi, g) \in [A, B]$$

Since $\int f\,dg$ $\exists$ we must have

$$\int_a^b f\,dg \in [A, B] \quad \text{too.} \qquad (\text{UB} \checkmark)$$

So $$I = \frac{\int_a^b f\,dg}{g(b) - g(a)} \in [\min f, \max f]$$



So by IVT for CTS f $\exists c$: $f(c) = I$ $\square$