Math 5302, Homework 1

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1. **[TK, Ex 8.1]** Suppose that the four desirable properties of area (Lecture 1, page 14) hold.
   (a) Let A be a bounded set in $\mathbb{R}^2$ and $B \subseteq A$. Show that $\mu(A) \geq \mu(B)$. [Hint: $A = B \cup (A \backslash B)$.]
   (b) Let A be a non-empty bounded open set in $\mathbb{R}^2$. Show that $\mu(A) > 0$.

2. **[TK, Ex 8.7]** Use the following deep theorem of Banach and Tarski to show that the four conditions below cannot all hold.

   Theorem 1. *The unit ball in $\mathbb{R}^3$ can be decomposed into a finite number of pieces which may be reassembled, using only translation and rotation, to form two disjoint copies of the unit ball.*

   (a) Every bounded set, E, in $\mathbb{R}^3$ has a volume, $\mu(E)$, with $\mu(E) \geq 0$.
   (b) Let E be a bounded set in $\mathbb{R}^3$ and let F be a rotation and translation of E. Then $\mu(F) = \mu(E)$.
   (c) If E is a cube of side length a then $\mu(E) = a^3$.
   (d) If E_1 and E_2 are disjoint bounded sets in $\mathbb{R}^3$, then $\mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2)$.

3. **[MP,5.2.6]** Show from the definition of the Riemann integral that if $\int_a^b f(x)\,dx$ exists then it is unique.

4. Prove that the Riemann integral on $\mathbb{R}$ is translation invariant in that if $f : [a, b] \to \mathbb{R}$ is Riemann integrable and $c \in \mathbb{R}$, then $g(x) = f(x - c)$ is Riemann integrable and
$$\int_{a+c}^{b+c} g(x)\,dx = \int_a^b f(x)\,dx.$$

5. Prove the converse of the Proposition on the Cauchy-like Criterion stated on the bottom of page 2 of Lecture 2. **Hint:** (1) Let P_n be the regular partition of $[a, b]$ with $n + 1$ points, and let ξ_n be any set of tags for P_n. Use the Cauchy Criterion for Riemann sums to prove that $I_n = \sigma(f, P_n, \xi_n)$ is a Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is complete, there is an I so that $I_n \to I$ as $n \to \infty$. So I is our candidate for the Riemann integral of f. (2) Now show that I is indeed the Riemann integral of f using the formal definition of Riemann integral. For that you will need to apply the triangle inequality and be very careful with your ϵ's and δ's.

Additional Problem [Not not hand in]

1. **[TK, Ex. 8.16]** Suppose that $f : [a, b] \to \mathbb{R}$ is Riemann integrable and $\lambda \in \mathbb{R}$. Prove that λf is Riemann integrable and that
$$\int_a^b \lambda f(x)\,dx \;=\; \lambda \int_a^b f(x)\,dx.$$