Math 152H, Fall 2008

Calculus II

Project Three: A Curious Sequence

1 Overview

In this project we study sequences defined by the recurrence relation

$$a_{n+1} = ka_n(1 - a_n). \tag{1}$$

Different choices of the parameter k and of the first term a_0 result in different sequences. This sequence has been used by ecologists to model the evolution of insect populations. In such a model, there is assumed to be a maximum population size (the "carrying capacity") that can be supported by the environment. The n-th term in the sequence, a_n, then represents the fraction of the maximum population that is alive in the n-th generation. When a_n is close to zero, $1 - a_n \approx 1$ and so equation (1) is approximately given by $a_{n+1} \approx ka_n$. In this case the population grows exponentially. However as a_n increases towards 1 the factor $1 - a_n$ gets closer to 0 which causes the population to decrease. In summary, the factor ka_n models the grow of population due to births (and deaths) when the population is not limited by the available resources and the factor $1 - a_n$ is included to account for the finite resources (and hence finite carrying capacity) of the environment.

The goal of the project is to study the different sorts of long term behaviour these sequences exhibit. Such an analysis helps ecologists answer questions like: Will the population stabilize to a limiting value? Will it change in a cyclical fashion? Under what circumstances does the population vary in a random or chaotic manner?

To study the properties of (1) it is useful to introduce the function

$$f(x) = kx(1 - x). \tag{2}$$

Then $a_{n+1} = f(a_n)$.

2 Questions

1. For ecology applications we need $0 \le a_n \le 1$ for all n. Suppose that $0 \le a_0 \le 1$. Use calculus to find the values of k for which $0 \le a_n \le 1$ for all positive integers n.

2. Write a computer program (preferably in Matlab) to calculate a table of values and graph the first N terms in the sequence (1). The input parameters to your program should be N, k and a_0. The output should be the table of values (n, a_n) together with the graph of a_n versus n. Note: In Matlab, to plot a sequence a_n versus n store the numbers n and a_n in arrays n and a and use the command *plot(n,a,'bx')*. This will put

a blue cross at each point $(n, a(n))$. Other colors are red (r), black (k), green (g) etc. Check your program works correctly by comparing its output to a pencil and paper calculation of the first few terms. Then use your program to answer the following questions.

3. Calculate about 25 terms of the sequence with $a_0 = 0.5$ for two values of k such that $1 < k < 3$. Graph the sequences. Do they appear to converge? If so, what do you think the limits are? Repeat for a different choice of a_0 with $0 < a_0 < 1$. Does the limit depend on the choice of a_0 or k or both?

4. Suppose $L = \lim_{n\to\infty} a_n$ exists. Do an algebraic calculation to determine the possible values of L. Compare your result to the graphs in the previous question.

5. For one of the k values you chose above plot the function in equation (2). On your plot graph the points (a_0, a_1), (a_1, a_2), (a_2, a_3),.... Join each point (a_n, a_{n+1}) to the next point with a line. (You can modify your matlab function to do this.) What happens to the points (a_n, a_{n+1}) as $n \to \infty$?

6. In general prove that if the sequences converges then the points (a_n, a_{n+1}) converge to an intersection point of the line $y = x$ with the graph of (2).

7. Now we are going to examine some different sorts of behaviour for different values of k. Calculate and graph terms in the sequence for a value of k between 3 and 3.4. What do you notice?

8. Experiment with values of k between 3.4 and 3.5. What happens to these sequences?

9. For values of k between 3.6 and 4, plot at least 100 terms and comment on the behaviour of the sequence. What happens if you change a_0 by 0.001? This type of behaviour is called chaotic and is exhibited by insect populations under certain conditions.

10. To get more of a feeling for the different sorts of behaviour we have observed in the examples above we are now going to study how the sequence depends on the first term a_0. For $k = 1, 3, 4$ and $n = 1, 2, 3, 4, 5$ plot a_n versus a_0 where $0 \le a_0 \le 1$. Another way to think about what I am asking you to do here is the following. For a fixed value of k we can think if a_1 as being a function of a_0. In fact this is just the function $a_1 = f(a_0)$, where f is given in (2). Similarly $a_2 = f(f(a_0))$ involves composition of f with itself, and so on. What do you learn from these graphs?

3 The Write Up

For your write up please include all calculations, any Matlab code you write, output and plots generated by Matlab, and a brief introduction and conclusion showing that you understand the point of the project.

Some additional points to keep in mind are:

1. The write up should be a self-contained, *i.e.*, it should make sense even if you don't have this Project Assignment Sheet.

2. Your audience is a fellow class mate who has studied the same material you have in class, but has not done this project.

3. Don't talk about your *experiences* doing the project. Rather, summarize your scientific/mathematical findings/results.

4. Your introduction should be in the form of an "executive summary" of the **results** you found, rather than simply being a restatement of the problem. For this be as concise and precise as you can!!

5. Be as precise as you can.

6. Make sure your figures have legends. Also, comment on what you learn from each figure.

7. *Do* **not** *assume that the author of this Project Assignment Sheet is infallible.* I may well have made a mistake!