NAME: SOLUTIONS

1 /12	2 /20	3 /15	4 /12 ~~/10~~	5 /10	6 /6 ~~/10~~	T /75

MATH 221H (Spring 2006) Exam 1, March 8th

No calculators, books or notes!
Show all work and give **complete explanations** for all your answers.
This is a 75 minute exam. It is worth a total of 75 points.

(1) [12 pts] Suppose that $\mathbf{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, and that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation such that $T(\mathbf{u}) = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ and $T(\mathbf{v}) = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$. Calculate $T\begin{pmatrix} 5 \\ -3 \end{pmatrix}$.

First find x_1, x_2 so that

$$\begin{pmatrix} 5 \\ -3 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The augmented matrix is

$$\left[\begin{array}{cc|c} \boxed{1} & 2 & 5 \\ 0 & \boxed{1} & -3 \end{array}\right]$$

which is already in row echelon form.

By back substitution, $x_2 = -3$ and $x_1 = 5 - 2x_2 = 11$

So $$\begin{pmatrix} 5 \\ -3 \end{pmatrix} = 11 \begin{pmatrix} 1 \\ 0 \end{pmatrix} - 3 \begin{pmatrix} 2 \\ 1 \end{pmatrix} \checkmark$$

Then by Linearity

$$T\left(\begin{pmatrix} 5 \\ -3 \end{pmatrix}\right) = 11\, T\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) - 3\, T\left(\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right)$$

$$= 11 \begin{pmatrix} 3 \\ -2 \end{pmatrix} - 3 \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 33 - 12 \\ -22 - 3 \end{pmatrix} = \begin{pmatrix} 21 \\ -25 \end{pmatrix}$$

(2) [20 pts] Consider the linear system

$$\begin{aligned} x - 2y + 2z - w &= 3 \\ 3x + y + 6z + 11w &= 16 \\ 2x - y + 4z + w &= 9. \end{aligned}$$

Calculate a row echelon form for this system. Then calculate a parametric vector form for the solution set.

$$\left[\begin{array}{cccc|c} \boxed{1} & -2 & 2 & -1 & 3 \\ 3 & 1 & 6 & 11 & 16 \\ 2 & -1 & 4 & 1 & 9 \end{array}\right] \quad \begin{array}{l} R2 \to R2 - 3R1 \\ R3 \to R3 - 2R1 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & -2 & 2 & -1 & 3 \\ 0 & 7 & 0 & 14 & 7 \\ 0 & 3 & 0 & 3 & 3 \end{array}\right] \quad \begin{array}{l} R2 \to \frac{1}{7} R2 \\ R3 \to \frac{1}{3} R3 \end{array}$$

Not strictly what one does in G.E. algorithm but simplies algebra + still gives row echelon form

$$\left[\begin{array}{cccc|c} \boxed{1} & -2 & 2 & -1 & 3 \\ 0 & \boxed{1} & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{array}\right] \quad R3 \to R3 - R2$$

$$\begin{array}{cccc} x & y & z & w \end{array}$$

$$\left[\begin{array}{cccc|c} \boxed{1} & -2 & 2 & -1 & 3 \\ 0 & \boxed{1} & 0 & 2 & 1 \\ 0 & 0 & 0 & \boxed{-1} & 0 \end{array}\right] \quad \text{Row Echelon Form.}$$

12

Back Sub^n

$w = 0$

z free

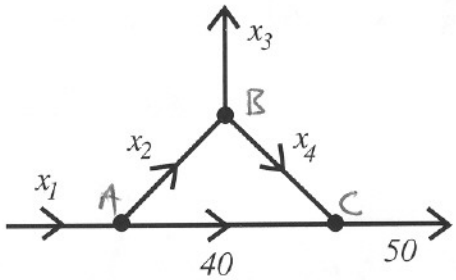
$y = 1 - 2w = 1 - 0 = 1$

$x = 3 + 2y - 2z + w = 3 + 2 - 2z = 5 - 2z$ 6

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 5 - 2z \\ 1 \\ z \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ 0 \\ 0 \end{pmatrix} + z \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \ \underline{\underline{z \in \mathbb{R}}}$$

is vector parametric form of solution 2 ✓

(3) [15 pts] Find the general flow pattern for the network shown in the figure. Assuming all the flows are nonnegative, find the smallest possible value for x_2?



FLOW CONSERVATION LAWS give us a system of linear equations:

TOTAL FLOW IN = TOTAL FLOW OUT : $x_1 = x_3 + 50$ (1)

FLOW INTO A = FLOW OUT OF A: $x_1 = x_2 + 40$ (2)

FLOW INTO B = FLOW OUT OF B: $x_2 = x_3 + x_4$ (3)

FLOW INTO C = FLOW OUT OF C : $40 + x_4 = 50$. (4) 6

From (4) can immediately see $\boxed{x_4 = 10}$.

So get system

$$\begin{aligned} x_1 - x_3 &= 50 \\ x_1 - x_2 &= 40 \\ x_2 - x_3 &= 10 \end{aligned} \quad \text{or} \quad \left[\begin{array}{ccc|c} \boxed{1} & 0 & -1 & 50 \\ 1 & -1 & 0 & 40 \\ 0 & 1 & -1 & 10 \end{array}\right] \quad R2 \to R2 - R1$$

$$\Leftrightarrow \left[\begin{array}{ccc|c} \boxed{1} & 0 & -1 & 50 \\ 0 & \boxed{-1} & 1 & -10 \\ 0 & 1 & -1 & 10 \end{array}\right] \quad R3 \to R3 + R2$$

$$\Leftrightarrow \left[\begin{array}{ccc|c} \boxed{1} & 0 & -1 & 50 \\ 0 & \boxed{-1} & 1 & -10 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Row Echelon Form.

So x_3 free

$$-x_2 = -10 - x_3$$

$$x_1 = 50 + x_3$$

GENERAL FLOW PATTERN IS:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 50 + x_3 \\ 10 + x_3 \\ x_3 \end{pmatrix} \quad x_4 = 10$$ 6

$x_3 \geq 0 \Rightarrow$ Smallest value for x_2 is 10. 3

(4) [~~10~~ 12 pts]

(a) Define what it means for a set of vectors to be linearly *independent.*

A set $\{\vec{v}_1 \ldots \vec{v}_n\}$ of n vectors in $\mathbb{R}^m$ is linearly independent if ~~the~~ whenever

$$c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = \vec{0}$$

the scalars c_j must all be zero: $c_1 = c_2 = \cdots = c_n = 0$. 4

(b) For what values of h are the following two vectors linearly *dependent*? Justify your answer.

$$\begin{pmatrix}1\\3\end{pmatrix}, \begin{pmatrix}4\\h\end{pmatrix}.$$

We need to find c_1, c_2 not both zero so that

$$c_1\begin{pmatrix}1\\3\end{pmatrix} + c_2\begin{pmatrix}4\\h\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$\begin{pmatrix}1 & 4\\3 & h\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \quad (*)$$

~~For~~ So we need a non-zero solution of the homogeneous system $(*)$. This will occur only when $\begin{pmatrix}1 & 4\\3 & h\end{pmatrix}$ has at least one free variable. 8

$$\left[\begin{array}{cc|c}1 & 4 & 0\\3 & h & 0\end{array}\right] \quad R2 \to R2 - 3R1$$

$$\Leftrightarrow \left[\begin{array}{cc|c}\boxed{1} & 4 & 0\\0 & h-12 & 0\end{array}\right] \quad \text{Row Echelon Form.}$$

So Ans is $h = 12$

$\boxed{\text{IF } h \neq 12}$ Then the $(1,1)$ and $(2,2)$ entries are PIVOT positions and so there are NO free variables when $h \neq 12$

$\boxed{\text{IF } h = 12}$ $\left[\begin{array}{cc|c}\boxed{1} & 4 & 0\\0 & 0 & 0\end{array}\right]$ c_2 is free: $-4\begin{pmatrix}1\\3\end{pmatrix} + 1\begin{pmatrix}4\\12\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$ so LDR

(5) [10 pts]

(a) Define what it means for a transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ to be linear.

We need

① $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$ for all $\vec{u}, \vec{v} \in \mathbb{R}^n$

and ② $T(c\vec{u}) = c\,T(\vec{u})$ for all $c \in \mathbb{R}$, and all $\vec{u} \in \mathbb{R}^n$. 2

(b) Which of the following are linear transformations? Explain!

(i) $T(x, y) = (3 + 5y, 4x - 9y)$

If T is a linear transformation we must have $T(\vec{0}) = \vec{0}$.

In this case $T(0,0) = (3,0) \neq (0,0)$ 4

So T cannot be linear.

(ii) $T(x, y) = (5x - y, 2x + 7y)$

$T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 5 & -1 \\ 2 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ is a matrix transformation. So it must be linear. 4

(6) [6 pts]
Suppose that $A = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3]$ is a 5×3 matrix and that the system $A\mathbf{x} = \mathbf{b}$ is consistent. Is $\mathbf{b}$ in $\text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$? Why?

Yes Since $A\vec{x} = \vec{b}$ is consistent

There is a solution $\vec{x}$ of $A\vec{x} = \vec{b}$.

So

$$\vec{b} = A\vec{x} = [\vec{a}_1 \quad \vec{a}_2 \quad \vec{a}_3]\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$= x_1 \vec{a}_1 + x_2 \vec{a}_2 + x_3 \vec{a}_3$$

$$\in \text{Span}\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$$

Pledge: *I have neither given nor received aid on this exam*

Signature: ______________________________