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NAME: SOLUTIONS

MATH 221H (Spring 2006) Exam 2, April 12th

No calculators, books or notes!
Show all work and give **complete explanations** for all your answers.
This is a 75 minute exam. It is worth a total of 75 points.

(1) [16 pts] Let A be the 3×3 matrix

$$A = \begin{pmatrix} 0 & 1 & 3 \\ 1 & 2 & 0 \\ 4 & 5 & 1 \end{pmatrix}.$$

(a) Calculate $\det(A)$ using a co-factor expansion.

$$\begin{vmatrix} 0 & 1 & 3 \\ 1 & 2 & 0 \\ 4 & 5 & 1 \end{vmatrix} \overset{\text{Row 1}}{=} -1 \begin{vmatrix} 1 & 0 \\ 4 & 1 \end{vmatrix} + 3 \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}$$

$$= -1(1-0) + 3(5-8)$$

$$= -1 - 9 = -10.$$

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) Calculate A^{-1} using elementary row operations.

$$[A|I] = \left[\begin{array}{ccc|ccc} 0 & 1 & 3 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 4 & 5 & 1 & 0 & 0 & 1 \end{array}\right] \quad \begin{array}{l} R2 \Leftrightarrow R1 \\ \text{THEN} \\ R3 \rightarrow R3 - 4R1 \end{array}$$

$$\Leftrightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 3 & 1 & 0 & 0 \\ 0 & -3 & 1 & 0 & -4 & 1 \end{array}\right] \quad R3 \rightarrow R3 + 3R2$$

$$\Leftrightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 3 & 1 & 0 & 0 \\ 0 & 0 & 10 & 3 & -4 & 1 \end{array}\right] \quad \begin{array}{l} R3 \rightarrow \frac{1}{10} R3 \\ \text{THEN} \quad R2 \rightarrow R2 - 3R3 \end{array}$$

$$\Leftrightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & \frac{1}{10} & \frac{12}{10} & -\frac{3}{10} \\ 0 & 0 & 1 & \frac{3}{10} & -\frac{4}{10} & \frac{1}{10} \end{array}\right] \quad R1 \rightarrow R1 - 2R2$$

$$\Leftrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2/10 & -14/10 & 6/10 \\ 0 & 1 & 0 & 1/10 & 12/10 & -3/10 \\ 0 & 0 & 1 & 3/10 & -4/10 & 1/10 \end{array}\right] \quad A^{-1} = \frac{1}{10}\begin{bmatrix} -2 & -14 & 6 \\ 1 & 12 & -3 \\ 3 & -4 & 1 \end{bmatrix}$$

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(c) Use the calculation you performed in (b) to find det(A).

1 Row Swap. and $R3 \rightarrow \frac{1}{10} R3$ means

$\det(A) = -10 \det(I) = -10.$

/4

X

[10 pts]
(a) Let $A = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5]$ be a 7×5 matrix. Suppose that the free variables are x_2 and x_4. Find a basis for the column space, Col(A), of A.

Col(A) is spanned by pivot cols of A. Since free variables are x_2 and x_4, pivot variables are x_1, x_3, x_5

So pivot cols are 1, 3, 5.

So $\mathrm{Col}(A) = \mathrm{Span}\{\vec{a}_1, \vec{a}_3, \vec{a}_5\}$

Note that pivot cols are linearly independent, so we get a basis!

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(b) Suppose B is a 5×3 matrix and that the general solution of the equation $B\mathbf{x} = \mathbf{0}$ is

$$\mathbf{x} = \begin{pmatrix} r + 2s \cancel{- 5t} \\ 5r + 7s \cancel{+ 3t} \\ 3r - 6s \cancel{+ 9t} \end{pmatrix}, \qquad \overset{r, s}{\cancel{s, t, r}} \in \mathbb{R}.$$

Find a basis for the nullspace, Nul(B), of B. Also, what is the dimension of Col(B)?

$$\vec{x} = r\begin{pmatrix}1\\5\\3\end{pmatrix} + s\begin{pmatrix}2\\7\\6\end{pmatrix} \cancel{+ t\begin{pmatrix}-5\\3\\9\end{pmatrix}}$$

$$\text{So } \mathrm{Nul}(B) = \mathrm{Span}\left\{\begin{pmatrix}1\\5\\3\end{pmatrix}, \begin{pmatrix}2\\7\\6\end{pmatrix}, \cancel{\begin{pmatrix}-5\\3\\9\end{pmatrix}}\right\}$$

As Neither is a multiple of the other.

Since these 2 vectors are linearly independent they form a basis for Nul(B).

$\dim \mathrm{Col}(B) + \dim \mathrm{Nul}(B) = \#\text{ Pivot Vars} + \#\text{ Free Vars} = \#\text{ Vars} = 3$

So $\dim \mathrm{Col}(B) = 3 - 2 = 1$

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X

) [15 pts] True or False? *Give a complete justification for your answers.*

(a) If there is a vector $y \in \mathbb{R}^n$ so that the equation $A\mathbf{x} = \mathbf{y}$ has more than one solution, then the columns of A span $\mathbb{R}^n$.

A is $n \times n$

FALSE

By assumption there are vectors $\vec{x}_1, \vec{x}_2$ with $\vec{x}_1 \neq \vec{x}_2$ and $A\vec{x}_1 = \vec{y}$, $A\vec{x}_2 = \vec{y}$.

So $A(\vec{x}_1 - \vec{x}_2) = A\vec{x}_1 - A\vec{x}_2 = \vec{y} - \vec{y} = \vec{0}$

So $\vec{x}_1 - \vec{x}_2$ is a nontrivial solution to $A\vec{x} = \vec{0}$

So by INV MATRIX THM, A is NOT INVERTIBLE and cols of A do not span $\mathbb{R}^2$

(b) The subset $H = \{(x, y) \in \mathbb{R}^2 \mid y = x^2\}$ of $\mathbb{R}^2$ is a vector subspace of $\mathbb{R}^2$.

✓ $0 \in H$ FALSE

. BUT $(2,4) \in H$ and $2 \in \mathbb{R}$ while $2(2,4) = (4,8) \notin H$.

So H is not closed under scalar multiplication

So H cannot be a subspace of $\mathbb{R}^2$

(c) Let A be a 5×6 matrix. Then the null space of A is a vector subspace of $\mathbb{R}^5$.

FALSE $Nul(A) = \{\vec{x} / A\vec{x} = \vec{0}\}$

We have $\underset{5\times 6}{A} \; \underset{6\times 1}{\vec{x}} = \underset{5\times 1}{\vec{0}}$

So $\vec{x} \in \mathbb{R}^6$ must hold

So Nul (A) is a subspace of $\mathbb{R}^6$ (NOT $\mathbb{R}^5$)

[12 pts]

(a) State the definition of a *basis* for a vector space.

A basis for a vector space V is a set of vectors $\{\vec{b}_1 - \vec{b}_n\}$ so that

① $\{\vec{b}_1 - \vec{b}_n\}$ is a linearly independent set

and ② $V = \text{Span}\{\vec{b}_1 \text{——} \vec{b}_n\}$

(b) Which of the following form a basis for $\mathbb{R}^3$? Why?

(i) $\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}2\\-1\\5\end{pmatrix}\right\}$ NO Every basis for $\mathbb{R}^3$ must contain 3 elements.

ALT The vector $\begin{pmatrix}1\\2\\3\end{pmatrix} \times \begin{pmatrix}2\\-1\\5\end{pmatrix}$ is perpendicular to plane spanned by $\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}2\\-1\\5\end{pmatrix}$ and hence is not in their span

(ii) $\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix}, \begin{pmatrix}6\\9\\12\end{pmatrix}\right\}$

NO This set is linearly dependent.

as $\begin{pmatrix}6\\9\\12\end{pmatrix} = 2\begin{pmatrix}1\\2\\3\end{pmatrix} + 1\begin{pmatrix}4\\5\\6\end{pmatrix}$

(iii) $\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix}, \begin{pmatrix}7\\8\\9\end{pmatrix}, \begin{pmatrix}10\\11\\12\end{pmatrix}\right\}$

No Every basis of $\mathbb{R}^3$ must contain 3 elements

(5) [12 pts]

(a) Suppose that A is an $n \times n$ matrix with $\det(A) = 3$. Find $\det(A^4)$ and $\det(5A)$.

$$\det(A^4) = \det(AAAA) = (\det A)^4 = 3^4$$

as $\det(AB) = \det(A)\det(B)$

$\det(5A) = 5^n \det(A) = 5^n \cdot 3$ as the row operations Row $i \to \frac{1}{5}$ Row i for $i = 1 - n$ convert $5A$ to A.

(b) Let P be the parallelogram determined by the vectors $\mathbf{a}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\mathbf{a}_2 = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$ and let $A = \begin{pmatrix} -1 & 3 \\ 2 & 5 \end{pmatrix}$. Find the area of the image of P under the linear transformation $T(\mathbf{x}) = A\mathbf{x}$.

$$\text{Area}(T(P)) = |\det(A)|\ \text{Area}(P)$$

$$= \left|\det\begin{pmatrix} -1 & 3 \\ 2 & 5 \end{pmatrix}\right| \times \left|\det\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}\right|$$

$$= |-5-6|\,|7-6| = 11$$

(b) Suppose S and T are the linear transformations given by

$$S((x_1, x_2, x_3)) = (3x_1 + 5x_2 - x_3, 4x_2 + 3x_3, x_1 - x_2 + 4x_3),$$

and $T(\mathbf{x}) = A\mathbf{x}$, where

$$A = \begin{pmatrix} 1 & 0 & 5 \\ 2 & -1 & 0 \\ 4 & 1 & 0 \end{pmatrix}.$$

Find the matrix C so that of $(S \circ T)(\mathbf{x}) = S(T(\mathbf{x})) = C\mathbf{x}$.

$$S\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 & 5 & -1 \\ 0 & 4 & 3 \\ 1 & -1 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_1 \\ x_3 \end{pmatrix} = B\vec{x}.$$

$$C\vec{x} = S(T(\vec{x})) = BA\vec{x}$$

$$C = BA = \begin{pmatrix} 3 & 5 & -1 \\ 0 & 4 & 3 \\ 1 & -1 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 & 5 \\ 2 & -1 & 0 \\ 4 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 9 & -6 & 15 \\ 20 & -1 & 0 \\ 15 & 5 & 5 \end{pmatrix}$$

(6) [10 pts]

(a) State the definition of what it means for a matrix A to be invertible.

An $n \times n$ matrix A is invertible if there is an $n \times n$ matrix C so that

$$AC = CA = I_n.$$

where $I_n = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$ is the $n \times n$ identity matrix

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(b) Using the definition you gave in (a) prove that if A is an $n \times n$ invertible matrix then for any vector $\mathbf{b} \in \mathbb{R}^n$ the equation $A\mathbf{x} = \mathbf{b}$ has a solution and that this solution is unique.

If $A\vec{x} = \vec{b}$ Then $CA\vec{x} = C\vec{b}$, $I\vec{x} = C\vec{b}$,

$\boxed{\vec{x} = C\vec{b}.}$ This is ~~the~~ a solution.

Lets check: $A(C\vec{b}) = AC\vec{b} = I\vec{b} = \vec{b}$ ✓

Suppose $\vec{x}_1$ and $\vec{x}_2$ and two solutions

Then $A\vec{x}_1 = \vec{b} = A\vec{x}_2$

$$A(\vec{x}_1 - \vec{x}_2) = \vec{0}$$

$$CA(\vec{x}_1 - \vec{x}_2) = \vec{0}$$

$$I(\vec{x}_1 - \vec{x}_2) = \vec{0}$$

$$\vec{x}_1 = \vec{x}_2.$$

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Pledge: *I have neither given nor received aid on this exam*

Signature: ______________________