NAME: SOLUTIONS

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MATH 251 (Spring 2008) Exam 2, Mar 31st

No calculators, books or notes! Show all work and give **complete explanations**.
This 65 minute exam is worth a total of 75 points.

(1) [15 pts]
(a) Find the curvature of the unit speed curve

$$\mathbf{r}(s) = (1+\cos(s/2), \sqrt{3}\cos(s/2), 2\sin(s/2)).$$

$$T(s) = \vec{r}\,'(s) = \left(-\tfrac{1}{2}\sin\left(\tfrac{s}{2}\right), -\tfrac{\sqrt{3}}{2}\sin\left(\tfrac{s}{2}\right), \cos s/2\right)$$

is the unit tangent vector.

The curvature is $K(s) = |T'(s)|$.

Well

$$T'(s) = \left(-\tfrac{1}{4}\cos\left(\tfrac{s}{2}\right), -\tfrac{\sqrt{3}}{4}\cos\left(\tfrac{s}{2}\right), -\tfrac{1}{2}\sin\left(\tfrac{s}{2}\right)\right)$$

So

$$K(s) = \sqrt{\left[\left(\tfrac{1}{4}\right)^2 + \left(\tfrac{\sqrt{3}}{4}\right)^2\right]\cos^2\left(\tfrac{s}{2}\right) + \tfrac{1}{4}\sin^2(s/2)}$$

$$= \sqrt{\tfrac{1}{4}\cos^2(s/2) + 1/4\sin^2(s/2)} = \tfrac{1}{2}$$

(b) The curve in (a) is a circle. What is the radius of this circle, and why?

The curvature of a circle is $K(s) = \frac{1}{r}$ where r is the radius of the circle.

Since $K(s) = \frac{1}{2}$ we conclude that $r = 2$

(2) [15 pts] Carefully sketch the level curves of the function $z = f(x, y) = x^2 - 4y^2$ at levels z
± 2. Each level curve should be labeled and all should be drawn to scale *on the same set of axes*

$\underline{z = 0}$ $\quad x^2 - 4y^2 = 0$

$$x = \pm 2y$$

$$y = \pm \tfrac{1}{2} x.$$

$\underline{z = 1}$ $\quad x^2 - 4y^2 = 1$

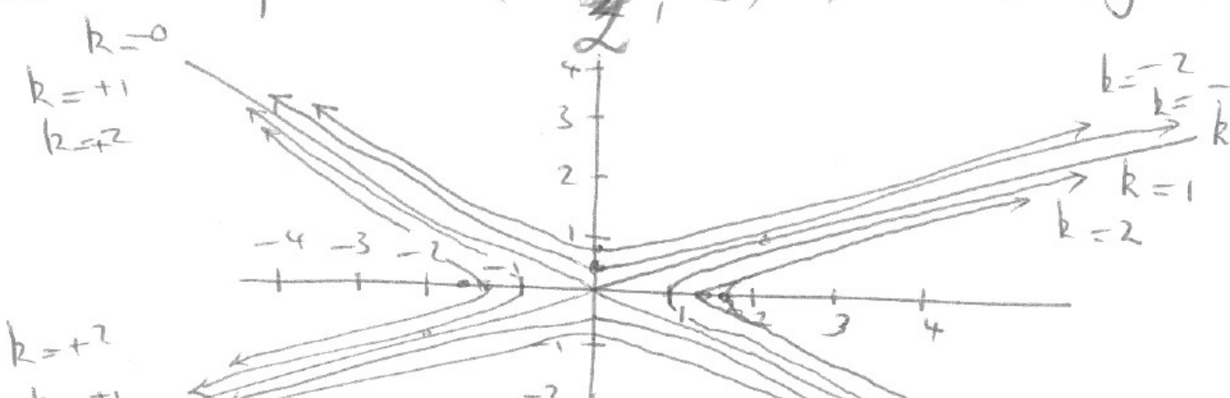
The points $(\pm 1, 0)$ lie on the curve which is a hyperbola that (by $z = 0$) asymptotes to the lines $y = \pm \frac{1}{2} x$.

$\underline{z = 2}$ The points $(\pm\sqrt{2}, 0)$ lie on $x^2 - 4y^2$

$\underline{z = -1}$ $\quad x^2 - 4y^2 = -1$ or $4y^2 - x^2 = 1$

The points $(\pm \frac{1}{2}, 0)$ lie on this hyperbol

$\underline{z = -2}$ The points $(\pm \frac{\sqrt{2}}{2}, 0)$ lie on $4y^2 -$



(3) [15 pts] Find all local maxima, local minima and saddle points of the function

$$z = f(x, y) = x^4 + y^4 - 4xy + 2.$$

Critical Points

$$0 = \frac{\partial f}{\partial x} = 4x^3 - 4y \implies y = x^3 \quad (1)$$

$$0 = \frac{\partial f}{\partial y} = 4y^3 - 4x \implies x = y^3 \quad (2)$$

Plug (2) into (1) : $y = y^9$

$$y(y^8 - 1) = 0$$

So $y = 0$ or $y^8 = 1$

$y = 0 \implies x = 0$

$y^8 = 1 \implies y = \pm 1 \implies x = y^3 = \pm 1$

So 3 critical points $(0,0)$, $(1,1)$, $(-1,-1)$.

$$D = \det \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{vmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{vmatrix} = 144x^2y$$

$\boxed{(0,0)}$ $D = -16 < 0$ So Saddle Point

$\boxed{(1,1)}$ $D = 144 - 16 > 0$ and $\frac{\partial^2 f}{\partial x^2} = 12 > 0$ So Local Min

$\boxed{(-1,-1)}$ $D = 144 - 16 > 0$ and $\frac{\partial^2 f}{\partial x^2} = 12 > 0$

(4) [20 pts]
(a) Show that the following limit does not exist:

$$\lim_{(x,y)\to(0,0)} \frac{xy\cos y}{x^2+3y^2}. \quad (*)$$

If we approach $(0,0)$ along the line $x=0$ we have

$$\lim_{y\to 0} \frac{0}{3y^2} = \lim_{y\to 0} 0 = 0$$

But if we approach $(0,0)$ along $y=x$ we have

$$\lim_{x\to 0} \frac{x^2\cos x}{x^2+3x^2} = \lim_{x\to 0} \frac{x^2\cos x}{4x^2} = \lim_{x\to 0} \frac{1}{4}\cos x$$

$= \frac{1}{4}$. Since the two limits are not equal $0 \neq \frac{1}{4}$ we conclude limit $(*)$ does not exist.

(b) Calculate the equation of the tangent plane to the function $z = f(x,y) = x^3e^{2y}$ at the point $(x,y,z) = (2,0,8)$.

The equation of the tangent plane to $z = f(x,y)$ at (a,b) is

$$z = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)$$

$f(2,0) = 8$

$$\frac{\partial f}{\partial x} = 3x^2e^{2y} \implies \frac{\partial f}{\partial x}(2,0) = 12$$

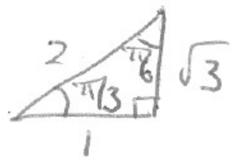
$$\frac{\partial f}{\partial y} = 2x^3e^{2y} \implies \frac{\partial f}{\partial y}(2,0) = 16$$

So equation of tangent plane is

$$z = 8 + 12(x-2) + 16(y-0)$$

(5) [10 pts] A mouse walks around a circle in the xy-plane. Suppose that the position of the mouse at time t is given by the parametrized curve $(x, y) = \mathbf{r}(t) = (\cos t, \sin t)$. Let $z = T(x, y)$ be the temperature function in the plane. Suppose that when the mouse is at the point $(x, y) = (\frac{\sqrt{3}}{2}, \frac{1}{2})$ it experiences a rate of change of temperature of 5 degrees Fahrenheit per second. Suppose that an ant is also walking at speed 1 centimeter per second, but that unlike the mouse it can walk whereever it wants to in the xy-plane. If the ant is at the same point $(\frac{\sqrt{3}}{2}, \frac{1}{2})$ as the mouse, in what direction should it walk to decrease the temperature T the fastest if $\frac{\partial T}{\partial x} = -2$ at $(\frac{\sqrt{3}}{2}, \frac{1}{2})$.

Let $g(t) = T(\vec{r}(t))$ be the temperature the ant experiences.



At $t = \pi/6$, $r(\pi/6) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

So we know $\boxed{g'(\pi/6) = 5}$

$\vec{r}(t) = (\cos t, \sin t$

$\vec{r}\,'(t) = (-\sin t, \cos$

By Chain Rule for Functions & Curves

$\vec{r}\,'(\pi/6) = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right.$

$$5 = g'(\pi/6) = \nabla T(\vec{r}(\pi/6)) \cdot \vec{r}\,'(\pi/6)$$

$$5 = \frac{\partial T}{\partial x}\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right) \frac{dx}{dt}\left(\frac{\pi}{6}\right) + \frac{\partial T}{\partial y}\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right) \frac{dy}{dt}(\pi/6)$$

$$5 = -2\left(-\frac{1}{2}\right) + \frac{\partial T}{\partial y}\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)$$

So $$\frac{\partial T}{\partial y}\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right) = \frac{2}{\sqrt{3}}(5-1) = \frac{8}{\sqrt{3}}$$

The ant should walk in direction

$$\vec{v} = -\nabla T(\sqrt{3}/2, \tfrac{1}{2}) \Big/ \left|\nabla T(\sqrt{3}/2, \tfrac{1}{2})\right| = \frac{(2, -8/\sqrt{3})}{\sqrt{4 + \frac{64}{3}}}.$$

Pledge: *I have neither given nor received aid on this exam*

Signature: ______________________________