NAME: SOLUTIONS

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MATH 423 (Spring 2004) Exam 1, March 8th

No calculators, books or notes!
Show all work and give **complete explanations** for all your answers.
This is a 75 minute exam. It is worth a total of 75 points.

(1) [15 pts]

/7 (a) Calculate the 1-form df, for the function $f(x,y) = x^2 + \cos(y)$, and evaluate $df(\mathbf{v}_p)$, where $p = (2, \pi/2)$, and $\mathbf{v} = (4,5)$.

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 2x\,dx - \sin y\,dy$$

$$df(\vec{v}_p) = 2.2.4 - \sin\left(\frac{\pi}{2}\right)5$$

$$= 16 - 5 = 11$$

/8 (b) Compute the covariant derivate $\nabla_V W$, of the vector field $W = (xy, x^2+y^3)$ on $\mathbf{R}^2$ with respect to the vector field $V = (y,x)$ at the point $p = (2,3)$.

$$\nabla_V W = \nabla_V (xy, x^2+y^3)$$

$$= (\vec{V}[xy], \vec{V}[x^2+y^3]) \qquad \vec{V}[f] = \nabla f \cdot V$$

$$= ((y,x)\cdot(y,x), (2x, 3y^2)\cdot(y,x))$$

$$= (x^2+y^2, 2xy + 3xy^2)$$

So $$\nabla_V W(2,3) = (2^2+3^2, 2.2.3 + 3.2.3^2)$$

$$= (13, 66)$$

(2) [15 pts] Let $\beta : I \to \mathbf{R}^3$ be the unit speed curve

$$\beta(s) \;=\; \left(3\cos(\tfrac{s}{5}),\, 3\sin(\tfrac{s}{5}),\, \tfrac{4}{5}s\right).$$

10/10 (a) Compute the Frenet frame and the curvature and torsion of β

$$T = \beta' = \left(-\frac{3}{5}\sin\left(\frac{s}{5}\right),\ \frac{3}{5}\cos\left(\frac{s}{5}\right),\ \frac{4}{5}\right)$$

$$T' = \left(-\frac{3}{25}\cos\left(\frac{s}{5}\right),\ -\frac{3}{25}\sin\left(\frac{s}{25}\right),\ 0\right) = \kappa N$$

So $\kappa = \|T'\| = 3/25$ $\qquad \boxed{\kappa = 3/25}$

$$N = T'/\kappa = \left(-\cos\left(\frac{s}{5}\right),\ -\sin\left(\frac{s}{5}\right),\ 0\right)$$

$$B = T \times N = \left(\frac{4}{5}\sin\frac{s}{5},\ -\frac{4}{5}\cos\frac{s}{5},\ \frac{3}{5}\right)$$

$B' = -\tau N$ gives us τ:

$$B' = \left(\frac{4}{25}\cos\frac{s}{5},\ \frac{4}{25}\sin\frac{s}{5},\ 0\right) = -\frac{4}{25}N$$

So $\boxed{\tau = \frac{4}{25}}$

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(b) Find the equation of the osculating plane of β at $s = 0$.

NORMAL TO PLANE

$$\vec{n} = B(0) = \left(0, -\frac{4}{5}, \frac{3}{5}\right)$$

POINT IN PLANE

$$\vec{x_0} = \beta(0) = (3, 0, 0)$$

EQUATION OF PLANE $\vec{x} = (x, y, z$

$$(\vec{x} - \vec{x_0}) \cdot \vec{n} = 0$$

OR

$$\boxed{4y = 3z}$$

(3) [15 pts] Let $\beta : I \to \mathbf{R}^3$ be a unit speed curve with curvature $\kappa > 0$.

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(a) Assuming that $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ are a frame, and that $\mathbf{T}' = \kappa\mathbf{N}$ and $\mathbf{B}' = -\tau\mathbf{N}$, prove that

$$\mathbf{N}' = -\kappa\mathbf{T} + \tau\mathbf{B}$$

Since T, N, B form a frame at each point on β and since N' is a vector field along β

$$N' = aT + bN + cB \quad \text{for some functions}$$

a, b, c on I.

Since T, N, B are mutually orthogonal and have length 1

$$N' = (N'\cdot T)\,T + (N'\cdot N)\,N + (N'\cdot B)\,B.$$

Also

① $N\cdot T = 0 \Rightarrow N'\cdot T + N\cdot T' = 0 \Rightarrow N'\cdot T = -T'\cdot N$
$= -\kappa N\cdot N$
$= -\kappa$

② $N\cdot N = 0 \Rightarrow 2N'\cdot N = 0 \Rightarrow N'\cdot N = 0$

③ $N\cdot B = 0 \Rightarrow N'\cdot B + N\cdot B' = 0 \Rightarrow N'\cdot B = -N\cdot B' = \tau N\cdot N = \tau$

So $N' = -\kappa T + \tau B$.

(b) Express $\mathbf{B}''$ in terms of $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$.

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$$B' = -\tau N$$

So $B'' = -(\tau N)' = -\tau' N - \tau N'$

$= -\tau' N - \tau(-\kappa T + \tau B)$ by ⓐ

$$B'' = \tau\kappa T - \tau' N - \tau^2 B$$

(4) [15 pts] One way to define a system of coordinates for the sphere S^2 given by

$$x^2 + y^2 + (z-1)^2 = 1$$

is to consider the stereographic projection mapping

$$\pi : S^2 \sim \{N\} \to \mathbf{R}^2$$

whose domain is the complement of the north pole $N = (0,0,2)$ in S^2. This mapping is defined as follows. Given a point p on $S^2 \sim \{N\}$, draw a line from the north pole, N, through the point p. The point $\pi(p)$ is the point where this line intersects the plane $z = 0$.

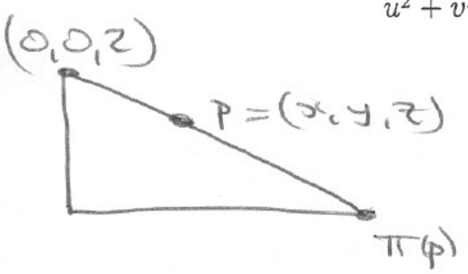
The stereographic coordinate patch is defined to be the function $\mathbf{x} : \mathbf{R}^2 \to S^2 \sim \{N\}$ given by the inverse function to π, i.e., $\mathbf{x} = \pi^{-1}$.

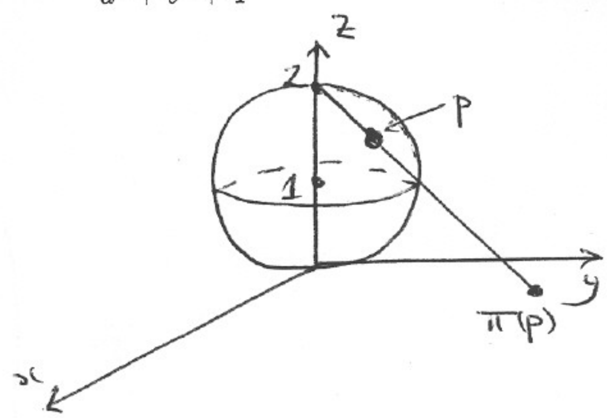
If $p = (x, y, z)$ and $\pi(p) = (u, v)$, then we have $\mathbf{x}(u, v) = (x, y, z)$.

/8

(a) Show that

$$x = \frac{4u}{u^2+v^2+4}, \qquad y = \frac{4v}{u^2+v^2+4}, \qquad z = \frac{2(u^2+v^2)}{u^2+v^2+4}$$





The line from (0,0,2) to (x, y, z) is parametrized by

$$\vec{l}(s) = (0\,0\,2) + s\,[(x\,y\,z) - (0\,0\,2)] = (sx,\ sy,\ 2 + s(z-2))$$

Find s so that $\vec{l}(s) = (u, v, 0)$:

$$2 + s(z-2) = 0 \quad \Rightarrow \quad \boxed{s = \frac{2}{2-z}}$$

~~Now~~ Then for this s:

$$x = \frac{u}{s}, \qquad y = \frac{v}{s}, \qquad z = 2\left(1 - \frac{1}{s}\right)$$

But $\quad x^2 + y^2 + (z-1)^2 = 1$

So $\quad 1 = \left(\frac{u}{s}\right)^2 + \left(\frac{v}{s}\right)^2 + \left(1 - \frac{z}{s}\right)^2$

$$1 = \frac{u^2+v^2 + (s-z)^2}{s^2}$$

$$\Rightarrow \boxed{s = \frac{u^2+v^2+4}{4}}$$

So $\quad x = \frac{u}{s} = \frac{4u}{u^2+v^2+4} \qquad y = \frac{v}{s} = \frac{4v}{u^2+v^2+4}$

$$z = 2\left(1 - \frac{1}{s}\right) = \frac{2(u^2+v^2)}{u^2+v^2+4}.$$

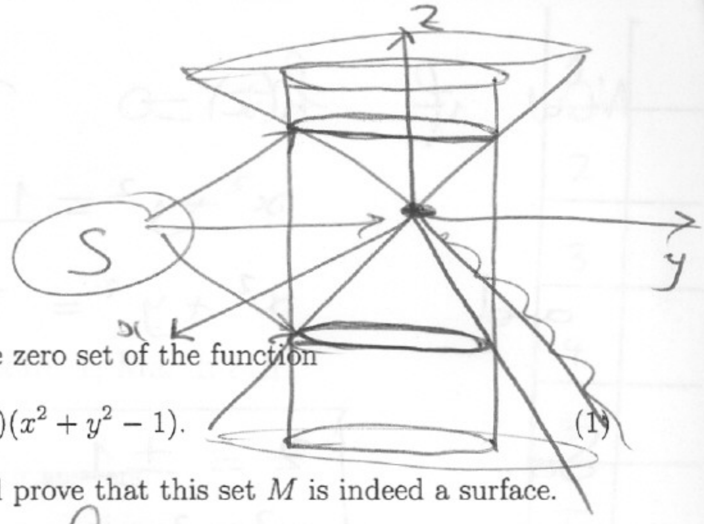
7 (b) Let $\mathbf{x}_* : T_{(u,v)}\mathbf{R}^2 \to T_{\mathbf{x}(u,v)}S^2$ be the tangent map of $\mathbf{x}$. Compute $\mathbf{x}_*(\mathbf{w}_{(u_0,v_0)})$ where $\mathbf{w} = (1,0)$ and $(u_0, v_0) = (1,2)$.

$$\vec{x}_*(\vec{w})_{(1,2)} = \frac{\partial \vec{x}}{\partial u}(1,2) \qquad \text{where } \vec{x} = (x, y, z)$$

NOW

$$\frac{\partial \vec{x}}{\partial u} = \begin{pmatrix} \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial u} \\ \frac{\partial z}{\partial u} \end{pmatrix} = \begin{pmatrix} \frac{4(v^2-u^2+4)}{(u^2+v^2+4)^2} \\ \frac{-8uv}{(u^2+v^2+4)^2} \\ \frac{16u}{(u^2+v^2+4)^2} \end{pmatrix}$$

So $\quad \frac{\partial \vec{x}}{\partial u}(1,2) = \left(\frac{28}{81}, -\frac{16}{81}, \frac{16}{81}\right)$



(5) [15 pts] Let $Z = \{(x, y, z) \in \mathbf{R}^3 \mid f(x, y, z) = 0\}$, be the zero set of the function

$$f(x, y, z) = (x^2 + y^2 - z^2)(x^2 + y^2 - 1). \tag{1}$$

Find the largest subset M of Z so that M is a surface, and prove that this set M is indeed a surface.

Let $g(x,y,z) = x^2+y^2-z^2$, $h(x,y,z) = x^2+y^2-1$.

Then $f = gh$

$Z = \text{Zero}(f) = \text{Zero}(g) \cup \text{Zero}(h)$ is the union of a double cone ($g=0$) and a cylinder ($h=0$).

We must find the points of Z at which $\nabla f = \vec{0}$, since these are the points at which Z is NOT a surface.

Let $S = \{(x\,y\,z) \mid f(x\,y\,z) = 0$ and $\nabla f(x\,y\,z) = \vec{0}\}$

Then $S = S_g \cup S_h$ where

$S_g = \{(x\,y\,z) \mid g(x\,y\,z) = 0$ and $\nabla f(x\,y\,z) = \vec{0}\}$

$S_h = \{(x\,y\,z) \mid h(x\,y\,z) = 0$ and $\nabla f(x\,y\,z) = \vec{0}\}$

NOW if $\vec{x} \in S_g$ then $g(\vec{x}) = 0$ and $\vec{0} = \nabla f(\vec{x}) = g(\vec{x})\nabla h(\vec{x}) + h(x)\nabla g(\vec{x})$

So $h(\vec{x})\nabla g(\vec{x}) = \vec{0}$

So $h(\vec{x}) = 0$ or $\nabla g(\vec{x}) = 0$.

PTO

Pledge: *I have neither given nor received aid on this exam*

Signature: ______________________

NOW if $h(\vec{x}) = 0$ then

$$x^2+y^2 = 1 \qquad (h=0)$$

and $$x^2+y^2 = z^2 \qquad (g=0) \quad \vec{x} \in S_g$$

So $\boxed{\begin{matrix} z = \pm 1 \\ x^2+y^2=1 \end{matrix}}$ Circles at height $z = \pm 1$ on both cylinder + cone.

If $\vec{0} = \nabla g(\vec{x}) = (2x, 2y, -2z)$

Then $(x,y,z) = (0,0,0)$ which is in S_g as $g(0,0,0) = 0$.

So
$$S_g = \{(x,y,z) \mid x^2+y^2 = 1 \text{ AND } z = \pm 1\} \cup \{(0,0,0)\}$$

<u>Also</u> $\vec{x} \in S_h \Rightarrow h(\vec{x}) = 0$ and

$$\vec{0} = \nabla f(\vec{x}) = g(\vec{x}) \nabla h(\vec{x}) + h(\vec{x}) \nabla g(\vec{x})$$
$$= g(\vec{x}) \nabla h(\vec{x})$$

SUMMARY: $M = Z \setminus S$
$S = \{(x,y,z) \mid x^2+y^2=1$ AND $z = \pm 1\} \cup \{(0,0,0)\}$.

So $g(\vec{x}) = 0$ or $\nabla h(\vec{x}) = \vec{0}$.

NOW $g(\vec{x}) = 0$ and $h(\vec{x}) = 0$ is $x^2+y^2=1$ and $z = \pm 1$ as above

$\nabla h(\vec{x}) = \vec{0} = (2x, 2y, 0)$ only at $(x,y,z) = (0,0,z)$ which is not in S_h.

~~SO $M = Z - S$, $S = \{(x,y,z)\}$~~