MATH 423 (Spring 2004) Exam 2, April 21st

No calculators, books or notes!
Show all work and give **complete explanations** for all your answers.
This is a 75 minute exam. It is worth a total of 80 points.

(1) [20 pts]
Let ϕ and ψ be the following forms on $\mathbf{R}^3$:

$$\begin{aligned} \phi &= x dx - y dy \\ \psi &= z dx \wedge dy + x dy \wedge dz. \end{aligned}$$

(a) Compute $\phi \wedge \psi$, $d\phi$ and $d\psi$.

$$\phi \wedge \psi = (x\,dx - y\,dy) \wedge (z\,dx \wedge dy + x\,dy \wedge dz)$$

$$= x^2\,dx \wedge dy \wedge dz \qquad \text{as } dx \wedge dx = 0 \text{ etc.}$$

$$d\phi = d(x\,dx - y\,dy) = dx \wedge dx - dy \wedge dy = 0$$

$$d\psi = d(z\,dx \wedge dy) + d(x\,dy \wedge dz)$$

$$= dz \wedge dx \wedge dy + dx \wedge dy \wedge dz$$

$$= dx \wedge dy \wedge dz + dx \wedge dy \wedge dz$$

$$= 2\,dx \wedge dy \wedge dz$$

(b) Compute $\iint_{\mathbf{x}} \psi$ where $\mathbf{x}$ is the patch for the cone $z^2 = x^2 + y^2$ defined by

$$\mathbf{x}(u,v) \;=\; (v\cos u, v\sin u, v), \qquad 0 \le u \le \tfrac{\pi}{2}, \qquad 0 < v < 2$$

$$\iint_{\vec{x}} \psi = \int_{u=0}^{u=\pi/2} \int_{v=0}^{v=2} \psi(\vec{x}_u, \vec{x}_v)\, dv\, du$$

Now

$$\vec{x}_u = (-v\sin u,\ v\cos u,\ 0)$$

$$\vec{x}_v = (\cos u,\ \sin u,\ 1)$$

$$\psi = z\, dx \wedge dy + x\, dy \wedge dz$$

$$\psi(\vec{x}_u, \vec{x}_v) = z\big(dx(\vec{x}_u)dy(\vec{x}_v) - dy(\vec{x}_u)dx(\vec{x}_v)\big)$$
$$+ x\big(dy(\vec{x}_u)dz(\vec{x}_v) - dz(\vec{x}_u)dy(\vec{x}_v)\big)$$

$$= v(-v\sin u \cdot \sin u - v\cos u\cos u)$$
$$+ v\cos u(v\cos u \cdot 1 - 0)$$

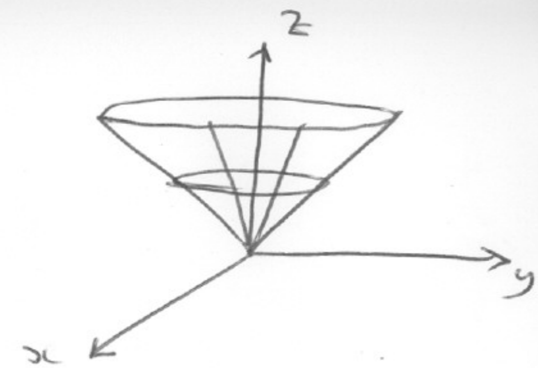
$\cos^2 u - 1 = -\sin^2 u$

$$= -v^2 + v^2\cos^2 u =$$

$$\iint_{\vec{x}} \psi = \int_{u=0}^{\pi/2} \int_{v=0}^{2} v^2(-\sin^2 u)\, dv\, du = -\int_0^2 v^2\, dv \int_0^{\pi/2} \sin^2 u\, du$$

$$= -\left[\frac{v^3}{3}\right]_0^2 \int_0^{\pi/2}\left(\frac{1}{2} - \frac{1}{2}\cos 2u\right)du = -\frac{8}{3}\,\frac{1}{2}\left[\frac{\pi}{2} - \left[\frac{\sin 2u}{2}\right]_0^{\pi/2}\right]$$

$$= \boxed{-8\pi/12} = \boxed{-2\pi/3}$$



(2) [20 pts] Let M be the conical surface $z^2 = x^2 + y^2$ with patch

$$\mathbf{x}(u,v) = (v\cos u, v\sin u, v), \qquad 0 \le u \le 2\pi, \qquad 0 < v < \infty.$$

(a) Compute the Gauss curvature K, mean curvature H, and principal curvatures of M. 16

(b) Sketch the cone M showing its principal curves. (You do not need to prove that the curves you draw are indeed the principal curves.) 4

$\vec{x}_u = (-v\sin u, v\cos u, 0)$

$\vec{x}_v = (\cos u, \sin u, 1)$

$\vec{x}_{uu} = (-v\cos u, -v\sin u, 0)$

$\vec{x}_{uv} = (-\sin u, \cos u, 0)$

$\vec{x}_{vv} = (0, 0, 0)$

$U = \dfrac{\vec{x}_u \times \vec{x}_v}{\|\vec{x}_u \times \vec{x}_v\|}$

$\vec{x}_u \times \vec{x}_v = \begin{vmatrix} i & j & k \\ -v\sin u & v\cos u & 0 \\ \cos u & \sin u & 1 \end{vmatrix}$

$= (v\cos u, v\sin u, -v)$

$\|\vec{x}_u \times \vec{x}_v\| = \sqrt{2}\, v$

$U = \frac{1}{\sqrt{2}}(\cos u, \sin u, -1)$

$E = \vec{x}_u \cdot \vec{x}_u = v^2$

$F = \vec{x}_u \cdot \vec{x}_v = 0$

$G = \vec{x}_v \cdot \vec{x}_v = 2.$

$L = U \cdot \vec{x}_{uu} = -\frac{1}{\sqrt{2}} v$

$M = U \cdot \vec{x}_{uv} = 0$

$N = U \cdot \vec{x}_{vv} = 0$

$K = \dfrac{LN - M^2}{EG - F^2} = 0 \qquad \boxed{K = 0}$

$H = \dfrac{GL + EN - 2FM}{2(EG - F^2)}$

$= \dfrac{-\frac{2}{\sqrt{2}} v + 0 + 0}{2(2v^2)}$

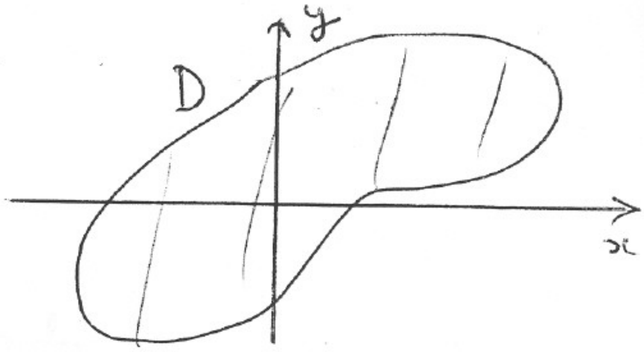
$\boxed{H = -\dfrac{\sqrt{2}}{4v}}$

$k_{1,2} = H \pm \sqrt{H^2 - K}$

$k_{1,2} = 0, 2H$

$k_1 = 0 \qquad k_2 = -\frac{\sqrt{2}}{2}\frac{1}{v}$

A TYPICAL SIMPLY CONNECTED REGION D



(3) [20 pts]

(a) Give a careful and complete statement of the Fundamental Theorem of Calculus for the case of 1-forms on surfaces. 6

(b) Let ω be the 1-form on $\mathbf{R}^2 \sim \{(0,0)\}$ defined by

$$\omega = \frac{1}{2\pi}\frac{-ydx + xdy}{x^2+y^2}$$

(i) Prove that $d\omega = 0$ on $\mathbf{R}^2 \sim \{(0,0)\}$. 4

(ii) Let $\alpha(t) = \epsilon(\cos t, \sin t)$, for $0 \le t \le 2\pi$, where $\epsilon > 0$ is a constant. Show that $\int_\alpha \omega = 1$. 4

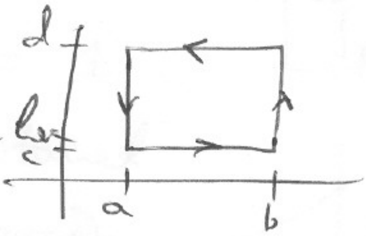
(iii) Let D be an open simply connected set in $\mathbf{R}^2$ that contains the origin, $(0,0) \in D$. [If a set is simply connected it doesn't have any holes]. Let β be a parametrization of the boundary of D. Show that $\int_\beta \omega = 1$. 6

(a) Let ϕ be a 1-form on M and let $\vec{x} : R \to M$ be a 2-segment in M. Then

$$\iint_{\vec{x}} d\phi = \int_{\partial\vec{x}} \phi$$

A 2-segment is a differentiable map $\vec{x} : R \to M$ from a rectangle $R = \{(x,y) \in \mathbb{R}^2 / a \le x \le b, c \le y \le d\}$ into a surface M.

The boundary $\partial\vec{x}$ of $\vec{x}$ is the oriented curve in M which is the image of the curve in $\mathbb{R}^2$ shown here



(3) (b) $\omega = \frac{+1}{2\pi} \; \frac{-y\,dx + x\,dy}{x^2+y^2}$

(i) $d\omega = \frac{1}{2\pi}\left[\frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) dy\wedge dx + \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) dx\wedge dy\right]$

$= \frac{1}{2\pi}\left[\frac{-1(x^2+y^2) + y.2y}{(x^2+y^2)^2}\, dy\wedge dx + \frac{1(x^2+y^2) - x.2x}{(x^2+y^2)^2}\, dx\wedge dy\right]$

$= \frac{1}{2\pi} \; \frac{+x^2+y^2 - 2y^2 + x^2+y^2-2x^2}{(x^2+y^2)^2}\, dx\wedge dy = 0$

(ii) $\alpha(t) = \varepsilon(\cos t, \sin t) = (x(t), y(t))$

$\alpha'(t) = \varepsilon(-\sin t, \cos t) = (x'(t), y'(t))$

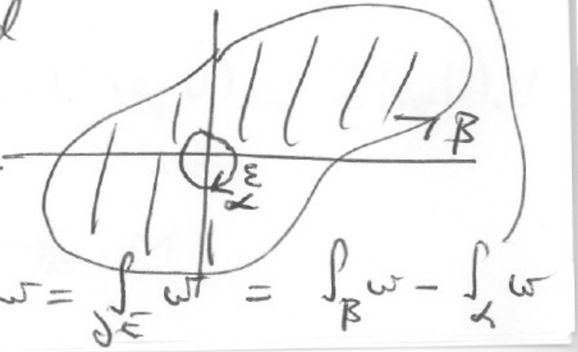
$\int_\alpha \omega = \int_0^{2\pi} \omega(\alpha'(t))\, dt = \frac{1}{2\pi}\int_0^{2\pi} \frac{-y(t)\,x'(t) + x(t)\,y'(t)}{x^2(t) + y^2(t)}\, dt$

$= \frac{1}{2\pi}\int_0^{2\pi} \frac{-\varepsilon^2 \sin t(-\sin t) + \varepsilon^2 \cos t \cos t}{\varepsilon^2(\cos^2 t + \sin^2 t)}\, dt$

$= \frac{1}{2\pi}\int_0^{2\pi} 1\, dt = 2\pi/2\pi = 1$

$\boxed{\text{So by (iii) } \int_\beta \omega = \int_\alpha \omega = 1.}$

(iii) LET E be the subset of $\mathbb{R}^2$ obtained by deleting a disk of radius ε ~~from D~~ about (0,0) from D, where ε is chosen so that disk is a subset of D.



Then $d\omega = 0$ on E. So by FTC $0 = \iint_E d\omega = \int_{\partial E} \omega = \int_\beta \omega - \int_\alpha \omega$

(4) [20 pts] Let M be a surface in $\mathbf{R}^3$.

6 (a) Define the shape operator S_p of M at a point p in M and prove that S_p is a linear map from the tangent space T_pM to itself, *i.e.*, $S_p : T_pM \to T_pM$.

2 (b) Define the normal curvature of M in direction $\mathbf{u}$ at p in terms of the shape operator.

12 (c) The maximum and minimum values of the normal curvature of M at p are called the *principal curvatures* of M, and are denoted k_1 and k_2. The directions in which these extremal values occur are called *principal directions* of M at p.

Prove that if p is not an umbilic point, then there are exactly two principal directions, that these directions are orthogonal, and that if $\mathbf{e}_1$ and $\mathbf{e}_2$ are unit vectors in the principal directions then

$$S(\mathbf{e}_1) = k_1\mathbf{e}_1 \qquad \text{and} \qquad S(\mathbf{e}_2) = k_2\mathbf{e}_2.$$

(a) Let $\vec{v} \in T_pM$. $S_p(\vec{v}) := -\nabla_{\vec{v}} U$ where U is a differentiable unit normal vector field on M.

- $S_p(a\vec{v} + b\vec{w}) = -\nabla_{a\vec{v}+b\vec{w}} U = -a\nabla_{\vec{v}} U - b\nabla_{\vec{w}} U$

$$= a\, S_p(\vec{v}) + b\, S_p(\vec{w})$$

- To show $S_p(\vec{v}) \in T_pM$ we must show

$$S_p(\vec{v}) \cdot U = 0 \quad , \quad \underline{\text{ie}} \quad (\nabla_{\vec{v}} U) \cdot U = 0$$

well $\quad 1 = U \cdot U$

$$\Rightarrow \quad 0 = \vec{v}[\vec{U} \cdot \vec{U}] = (\nabla_{\vec{v}} U) \cdot U + U \cdot (\nabla_{\vec{v}} U) = 2(\nabla_{\vec{v}} U) \cdot U$$

So $(\nabla_{\vec{v}} U) \cdot U = 0$.

(b) Let $\vec{u} \in T_pM$ be a unit tangent vector to M.

$$k(\vec{u}) = S_p(\vec{u}) \cdot \vec{u}.$$

(c) Fix p. Let $k_1 = \max_{\substack{\vec{u} \in T_pM \\ |\vec{u}|=1}} k(\vec{u})$

(4) (b) $\vec{e}_1 = \operatorname{argmax}_{\substack{\vec{u} \in T_pM \\ |\vec{u}|=1}} k(\vec{u})$

Let $\vec{e}_2 \in T_pM$ be a unit vector orthogonal to $\vec{e}_1$

For any $\vec{u} \in T_pM$ $\exists\, \theta \in [0, 2\pi)$: $\vec{u} = \vec{u}(\theta) = \cos\theta\, \vec{e}_1 + \sin\theta\, \vec{e}_2$

So $k = k(\theta) = k(\vec{u}(\theta)) = S(\cos\theta\, \vec{e}_1 + \sin\theta\, \vec{e}_2) \cdot (\cos\theta\, \vec{e}_1 + \sin\theta\, \vec{e}_2)$

Let $S_{ij} = S(\vec{e}_i) \cdot \vec{e}_j$ ~~i,j = 1,2~~ $1 \le i, j \le 2$.

Then

$k(\theta) = S_{11} \cos^2\theta + 2 S_{12} \cos\theta \sin\theta + S_{22} \sin^2\theta$ by linearity + symmetry of S.

We must find θ that are critical points of k.

$$\frac{dk}{d\theta} = 2 \sin\theta \cos\theta (S_{22} - S_{11}) + 2(\cos^2\theta - \sin^2\theta) S_{12}.$$

$\boxed{\text{IF } \theta = 0}$ Then $\vec{u}(0) = \vec{e}_1$, and so by assumpn $k(0)$ is a MAX

So $\frac{dk}{d\theta} = 0 = 0 + 2S_{12} \Rightarrow \boxed{S_{12} = 0}$

NOW since $\vec{e}_1, \vec{e}_2$ is ONB for T_pM, $S(\vec{e}_1) = S_{11} \vec{e}_1$

$S(\vec{e}_2) = S_{22} \vec{e}_2$

as $S_{12} = 0$. (PTO)

Pledge: *I have neither given nor received aid on this exam*

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So $S_{11} = k(\vec{e}_1) = k_1$ holds

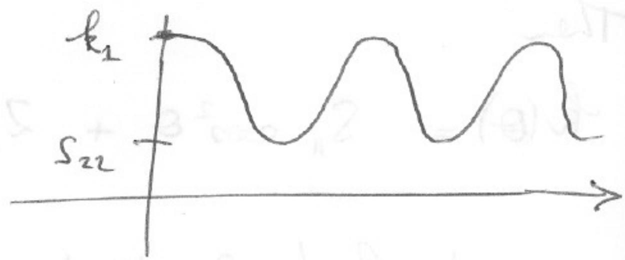
NOW $k(\theta) = k_1 \cos^2\theta + S_{22} \sin^2\theta$

Since p is not an umbilic point, $k(\theta)$ is not constant, $(k_1 \neq S_{22})$ and as k_1 is the max of $k(\theta)$ we must have $k_1 > S_{22}$.

So $k(\theta) = S_{22} \sin^2\theta + S_{22} \cos^2\theta + (k_1 - S_{22}) \cos^2\theta$

$$= \underbrace{(k_1 - S_{22})}_{>0} \cos^2\theta + S_{22}$$

has its min when



$\cos^2\theta = 0$

ie $\theta = \pi/2,\ 3\pi/2$

that is when $\vec{u} = \pm \vec{e}_2$.

Therefore $S(\vec{e}_2) = S_{22}\, \vec{e}_2 \Rightarrow k_2 = S_{22}$ holds.