1 /12	2 /10	3 /25	4 /10	5 /18	T /75

NAME: SOLUTIONS

MATH 423 (Spring 2006) Exam 1, March 8th

No calculators, books or notes!
Show all work and give **complete explanations** for all your answers.
This is a 75 minute exam. It is worth a total of 75 points.

(1) [12 pts]
(a) Let V and W be the vector fields on $\mathbf{R}^3$ defined by $V = x^2U_1 + xyU_2$ and $W = xy^2U_1 + (x^3+y)U_2$. Calculate the covariant derivative $\nabla_V W$. [Recall that $U_1 = (1,0,0)$, $U_2 = (0,1,0)$, and $U_3 = (0,0,1)$.]

$$\nabla_V W = V[xy^2]U_1 + V[x^3+y]U_2$$

$$= (x^2U_1 + xyU_2)[xy^2]U_1 + (x^2U_1 + xyU_2)[x^3+y]U_2$$

$$= \left(x^2\frac{\partial}{\partial x}(xy^2) + xy\frac{\partial}{\partial y}(xy^2)\right)U_1 + \left(x^2\frac{\partial}{\partial x}(x^3+y) + xy\frac{\partial}{\partial y}(x^3+y)\right)U_2$$

$$= (x^2y^2 + xy\,2xy)U_1 + (x^2\,3x^2 + xy\cdot 1)U_2$$

$$= 3x^2y^2U_1 + (3x^4+xy)U_2$$

(2) [10 pts]
(a) Define the direction derivative of a function $f : \mathbf{R}^3 \to \mathbf{R}$ in direction $\mathbf{v} \in T_p\mathbf{R}^3$ at a point $p \in \mathbf{R}^3$.

$$\vec{v}[f] := \frac{d}{dt}\Big|_{t=0} \left(f(\vec{p} + t\vec{v}) \right)$$

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(b) Prove that if $\beta : \mathbf{R} \to \mathbf{R}^3$ is *any* curve with $\beta(0) = p$ and $\beta'(0) = \mathbf{v}$, then $\mathbf{v}[f] = (f \circ \beta)'(0)$.

$$(f \circ \beta)'(0) \overset{CR}{=} \nabla f(\beta(0)) \cdot \beta'(0)$$

$$= \nabla f(p) \cdot \vec{v}$$

$$= \nabla f(\alpha(0)) \cdot \alpha'(0)$$

where $\alpha(t) = \vec{p} + t\vec{v}$

$$\overset{CR}{=} (f \circ \alpha)'(0)$$

$$= \frac{d}{dt}\Big|_{t=0} \left(f(\vec{p} + t\vec{v}) \right)$$

$$= \vec{v}[f]$$

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(3) [25 pts]

(a) State the definition of the tangent map, F_*, of a mapping $F : \mathbf{R}^n \to \mathbf{R}^m$.

Let $\vec{p} \in \mathbb{R}^n$. Define $F_* : T_p\mathbb{R}^n \longrightarrow T_{F(p)}\mathbb{R}^m$ as follows.

Let $\vec{v} \in T_p\mathbb{R}^n$ and $\alpha : I \to \mathbb{R}^n$ be the curve $\alpha(t) = \vec{p} + t\vec{v}$.

Define $\beta : I \to \mathbb{R}^m$ by $\beta(t) = (F \circ \alpha)(t) = F(\alpha(t))$.

Then

$$F_*(\vec{v}) := \beta'(0) = (F\circ\alpha)'(0) \in T_{F(p)}\mathbb{R}^m$$

is a tangent vector at $F(p)$, since $\beta(0) = F(p)$.

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Parts (b), (c), (d) below all concern the mapping $F : \mathbf{R}^2 \to \mathbf{R}^2$ defined by

$$F(u,v) = (u^2 - v^2, u^2 + v^2).$$

(b) Use the definition in (a) above to calculate $F_*(\mathbf{v})$, where $\mathbf{v} = (3,7)$ is the tangent vector to $\mathbf{R}^2$ at $p = (1,2)$.

$$\alpha(t) = (1,2) + t(3,7) = ((1+3t), (2+7t))$$

$$\beta(t) = F(\alpha(t)) = F(1+3t,\ 2+7t)$$

$$\beta(t) = ((1+3t)^2 - (2+7t)^2,\ (1+3t)^2 + (2+7t)^2)$$

$$\beta'(t) = (2(1+3t)3 - 2(2+7t)7,\ 2(1+3t)3 + 2(2+7t)7)$$

$$F_*(\vec{v}) = \beta'(0) = (6 - 28,\ 6 + 28)$$

$$= (-22, 34)$$

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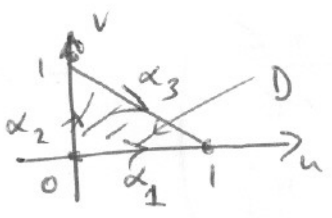
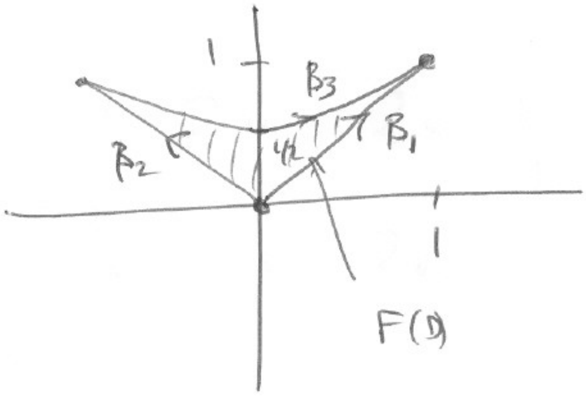
(c) Let D be the triangle in $\mathbf{R}^2$ whose vertices are at $(0,0)$, $(1,0)$, and $(0,1)$. Find the image $F(D)$ of D under the mapping F and carefully sketch it.





① $\alpha_1(t) = (t, 0) \qquad 0 \le t \le 1$

$\beta_1(t) = F(\alpha_1(t)) = F(t, 0) = (t^2, t^2)$

So β_1 lies on line $y = x$ and goes from $(0,0)$ to $(1,1)$

② $\alpha_2(t) = (0, t), \qquad 0 \le t \le 1$

$\beta_2(t) = F(\alpha_2(t)) = F(0,t) = (-t^2, t^2)$

So β_2 lies on line $y = -x$ from $(0,0)$ to $(-1,1)$

③ $\alpha_3(t) = (t, 1-t) \qquad 0 \le t \le 1$

$\beta_3(t) = F(t, 1-t) = (t^2 - (1-t)^2, t^2 + (1-t)^2)$

$= (t^2 - 1 + 2t - t^2, t^2 + 1 - 2t + t^2)$

$\beta_3(t) = (-1 + 2t, 1 - 2t + 2t^2) = (x, y)$

$x = -1 + 2t \qquad y = 1 - 2t + 2t^2 = -x + 2\left(\frac{x+1}{2}\right)^2$ 7

$t = \frac{x+1}{2} \qquad y = -x + \frac{x^2 + 2x + 1}{2} = \frac{x^2}{2} + 1/2$

(d) Calculate the matrix of partial derivatives of F and use it to find the set of points $(u, v) \in \mathbf{R}^2$ at which the tangent mapping F_* is 1-1 and onto.

$$DF = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} 2u & 2u \\ -2v & 2v \end{pmatrix}$$

$\det(DF) = 8uv = 0$ at $u = 0$ or $v = 0$

So F_* is 1-1 and ONTO where

$u \neq 0$ and $v \neq 0$

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(4) [10 pts]

(a) State the Frenet formulae for a unit speed curve.

$$T' = \kappa N$$

$$N' = -\kappa T + \tau B$$

$$B' = -\tau N$$

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(b) Prove that $\mathbf{N}'' \bullet \mathbf{N} < 0$.

$$N'' = (-\kappa T + \tau B)'$$

$$= -\kappa' T + -\kappa T' + \tau' B + \tau B'$$

$$= -\kappa' T - \kappa^2 N + \tau' B - \tau^2 N$$

$$= -\kappa' T - (\kappa^2 + \tau^2) N + \tau' B$$

So $N'' \cdot N = -(\kappa^2 + \tau^2) < 0$

$\boxed{\text{OR}}$

$$N' \cdot N = (\kappa T + \tau B) \cdot N = 0$$

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So

$$0 = (N' \cdot N)' = N'' \cdot N + N' \cdot N'$$

$$N'' \cdot N = -N' \cdot N' = -\|N'\|^2 < 0.$$

(5) [18 pts]
Let $\beta : I \to \mathbf{R}^3$ be the unit speed curve

$$\beta(s) = \left(3\cos(\tfrac{s}{5}), 3\sin(\tfrac{s}{5}), \tfrac{4}{5}s\right).$$

(a) Compute the Frenet frame and the curvature and torsion of β. [More space is provided on next page!]

$$T(s) = \beta'(s) = \left(-\frac{3}{5}\sin\left(\frac{s}{5}\right), \frac{3}{5}\cos\left(\frac{s}{5}\right), \frac{4}{5}\right)$$ 2

$$T' = \left(-\frac{3}{25}\cos\left(\frac{s}{5}\right), -\frac{3}{25}\sin\left(\frac{s}{5}\right), 0\right) = \kappa N$$

So $\kappa = \frac{3}{25} = \|T'\|$ 3

and

$$N = T'/\kappa = \left(-\cos(s/5), -\sin(s/5), 0\right)$$ 2

$$B = T \times N = \left(4/5 \sin s/5, -4/5 \cos s/5, 3/5\right)$$ 2

$B' = -\tau N$ gives us τ:

$$B' = \left(4/25 \cos s/5, \frac{4}{25}\sin s/5, 0\right) = -\frac{4}{25}N$$

So $\tau = 4/25$

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(b) Find the equation of the osculating plane of β at $s = 5\pi/2$.

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Normal to plae is

$$\vec{n} = B(5\pi/2) = \left(\frac{4}{5}\sin\left(\frac{5\pi}{2.5}\right), -\frac{4}{5}\cos(\pi/2), 3/5\right)$$

$$= \left(\frac{4}{5}, 0, 3/5\right)$$

Point in plae is $\vec{x}_0 = \beta(5\pi/2) = \left(0, 3, \frac{4\pi}{2}\right) = (0, 3, 2\pi)$

So eqn is

$$(\vec{x} - \vec{x}_0) \cdot \vec{n} = 0$$

$$(x, y-3, z-2\pi) \cdot (4/5, 0, 3/5) = 0$$

$$\boxed{4/5\, x + 3/5\,(z - 2\pi) = 0}$$

Pledge: *I have neither given nor received aid on this exam*

Signature: ______________________