NAME: SOLUTIONS

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| 1 /15 | 2 /10 | 3 /20 | 4 /15 | 5 /15 | T /75 |

MATH 423 (Spring 2006) Exam 2, April 24th

No calculators, books or notes!
Show all work and give **complete explanations** for all your answers.
This is a 75 minute exam. It is worth a total of 75 points.

(1) [15 pts] *Do not simplify your answers!*
Let $\omega = xyz\,dx + \sin(xy)\,dy + (x^2+z^3)\,dz$ and $\eta = z\,dx + e^{xz}dy$ be 1-forms on $\mathbf{R}^3$.

(a) Calculate $\omega(\mathbf{u}_p)$, where $p = (1,2,3)$ and $\mathbf{u} = (4,5,6)$.

$dx(\vec{u}) = 4, \quad dy(\vec{u}) = 5, \quad dz(\vec{u}) = 6$

So

$\omega(\vec{u}_p) = 1.2.3.4 + \sin(1.2)5 + (1^2+3^3)6$

(b) Calculate $\omega \wedge \eta$ in terms of $dx \wedge dy$, $dx \wedge dz$ and $dy \wedge dz$.

$\omega \wedge \eta = (xyz\,dx + \sin(xy)dy + (x^2+z^3)dz) \wedge (z\,dx + e^{xz}dy)$

$= xyz\,e^{xz}\,dx\wedge dy + z\sin(xy)\,dy\wedge dx$

$+ z(x^2+z^3)\,dz\wedge dx + e^{xz}(x^2+z^3)\,dz\wedge dy$

$= [xyz\,e^{xz} - z\sin(xy)]\,dx\wedge dy$

$- z(x^2+z^3)\,dx\wedge dz$

$- e^{xz}(x^2+z^3)\,dy\wedge dz$

(c) Calculate $d\eta$ and then evaluate $d\eta(\mathbf{v}_p, \mathbf{w}_p)$, where $p = (1, 0, 2)$, $\mathbf{v} = (4, -1, 0)$ and $\mathbf{w} = (2, 5, 3)$.

$$\boxed{(\omega_1 \wedge \omega_2)(\vec{v}_1, \vec{v}_2) = \omega_1(\vec{v}_1)\,\omega_2(\vec{v}_2) - \omega_1(\vec{v}_2)\,\omega_2(\vec{v}_1)}$$

$$\eta = z\,dx + e^{xz}\,dy$$

$$d\eta = dz \wedge dx + d(e^{xz}) \wedge dy$$

$$= dz \wedge dx + z e^{xz}\,dx \wedge dy + x e^{xz}\,dz \wedge dy$$

$$d\eta(\vec{v}_p, \vec{w}_p) = dz(\vec{v}_p)\,dx(\vec{w}_p) - dz(\vec{w}_p)\,dx(\vec{v}_p) + \ldots\ldots$$

$$= 0.2 - 3.4 + 2 \cdot e^{1.2}(4.5 - 2(-1)) + 1 \cdot e^{1.2}(0.5 - 3(-1))$$

(2) [10 pts]

(a) State the definition of a 2-form on a surface $M$.

A 2-form $\omega$ on a surface $M$ is a function that at each point $p \in M$ assigns a real number $\omega(\vec{v}_p, \vec{w}_p)$ to each ordered pair of tangent vectors $\vec{v}_p, \vec{w}_p \in T_pM$ so that

1. $\omega$ is linear in both $\vec{v}, \vec{w}$
2. $\omega(\vec{v}_p, \vec{w}_p) = -\omega(\vec{w}_p, \vec{v}_p)$

(b) Explain how a vector field on a surface $M$ can be used to define a 2-form on $M$.

Let $V$ be a vector field on $M$, i.e. that for each $p \in M$, $V(p) \in T_p\mathbb{R}^3$. Define a 2-form $\omega_V$ on $M$ by

$$\omega_V(\vec{v}_p, \vec{w}_p) = V(p) \cdot (\vec{v}_p \times \vec{w}_p).$$

The two properties in (2a) above hold for $\omega_V$ since dot and cross products are linear in both arguments and as $\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$.

(3) [20 pts]
(a) Let $\mathbf{x} : D \to \mathbf{R}^3$ be the function given by $\mathbf{x}(\theta, r) = (r\cos\theta, r^2, r\sin\theta)$, where $D$ is the region of the $(\theta, r)$-plane given by $0 < \theta < 2\pi$ and $r > 0$. Show that $\mathbf{x}$ is a proper patch for the paraboloidal surface $y = x^2 + z^2$ with normal in the positive-$y$ direction.

A proper patch for M is a function $\vec{x} : D \to \mathbb{R}^3$ so that

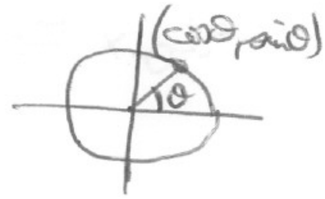
1. $\boxed{\vec{x} \text{ is 1-1}}$ $\boxed{\text{and } \vec{x}^{-1} \text{ is CONTINUOUS}}$

Suppose $\vec{x}(\theta_1, r_1) = \vec{x}(\theta_2, r_2)$. We must show $\theta_1 = \theta_2$, $r_1 = r_2$. (Here $(\theta_j, r_j) \in D$ for $j = 1, 2$).

Well $r_1 \cos\theta_1 = r_2 \cos\theta_2$; $r_1^2 = r_2^2$, $r_1 \sin\theta_1 = r_1 \sin\theta_2$.

By 2nd eqn, $r_1 = r_2$ as $r_j > 0$.

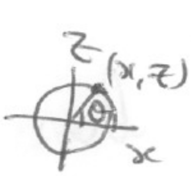
Then by 1st + 3rd eqns, $\cos\theta_1 = \cos\theta_2$
$\sin\theta_1 = \sin\theta_2$




Hence $\theta_1 = \theta_2$ must hold as $\theta_j \in (0, 2\pi)$.

So $\vec{x}$ is 1-1.

Since $x = r\cos\theta$, $y = r^2$, $z = r\sin\theta$, we can solve for $r, \theta$ to get $r = \sqrt{y}$, $\theta =$ Angle:




So $(\theta, r) = \vec{x}^{-1}(x, y, z) = (\text{Angle } \cancel{\text{for}} \text{ of } (x,z), \sqrt{y})$ is CTS.

2. $\boxed{\vec{x} \text{ is Regular.}}$

$$D\vec{x} = [\vec{x}_\theta, \vec{x}_r] = \begin{bmatrix} -r\sin\theta & \cos\theta \\ 0 & 2r \\ -r\cos\theta & \sin\theta \end{bmatrix}$$ which has full-rank 2 12

as the 2 columns are non-zero vectors which are not multiples of each other. Hence they are LI.

PTO

(3) $\boxed{\vec{x}(D) \subset M}$ where $M$ is paraboloid.

Well if $(x, y, z) = \vec{x}(\theta, r)$ Then

$$x^2 + z^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2 = y.$$

So $\vec{x}(D) \subset M$ ✓.

(4) $\vec{x}_\theta \times \vec{x}_r$ must have positive $y$-component, so that orientation induced by $\vec{x}$ on $M$ is correct.

Well

$$\vec{x}_\theta \times \vec{x}_r = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -r\sin\theta & 0 & r\cos\theta \\ \cos\theta & 2r & \sin\theta \end{vmatrix}$$

$$= (-2r^2\cos\theta, +r, -2r^2\sin\theta)$$

has a positive $y$ component as $r > 0$ ✓

$$\int_{\theta=a}^{b} \int_{\phi=c}^{d}$$

(b) Let $M$ be the part of the paraboloid $y = x^2 + z^2$ that lies inside the cylinder $x^2 + z^2 = 4$ with normal in the positive-$y$ direction. Let $\omega = xdy \wedge dz + z^2 dx \wedge dy$. Express $\int_M \omega$ in the form $\iint_D f(\theta, r)\, dr d\theta$. Find an explicit formula for the function $f$ in terms of $\theta$ and $r$, but do not attempt to evaluate the integral.

$$\int_M \omega \stackrel{\text{DEF}}{=} \int_{r=0}^{2} \int_{\theta=0}^{2\pi} \omega(\vec{x}_\theta, \vec{x}_r)\, d\theta\, dr$$

$$\vec{x}(\theta, r) = (x, y, z) = (r\cos\theta, r^2, r\sin\theta)$$

So $f(\theta, r) = \omega(\vec{x}_\theta, \vec{x}_r)$

$$= (x\,dy \wedge dz + z^2\,dx \wedge dy)\big((-r\sin\theta, 0, r\cos\theta), (\cos\theta, 2r, \sin\theta)\big)$$

$$= r\cos\theta\,(0 \cdot \sin\theta - 2r\, r\cos\theta)$$

$$+ r^2\sin^2\theta\,(-r\sin\theta \cdot 2r - \cos\theta \cdot 0)$$

$$= -2r^3\cos^2\theta - 2r^4\sin^3\theta$$

So

$$\int_M \omega = \int_{\theta=0}^{2\pi} \int_{r=0}^{2} (-2r^3\cos^2\theta - 2r^4\sin^3\theta)\, dr\, d\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{2} -2r^3\cos^2\theta\, dr\, d\theta$$

as $\int_0^{2\pi} \sin^3\theta = 0$ by symmetry.

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(4) [15 pts]
(a) State the definition of the shape operator, $S$, of an oriented surface $M$.

Let $p \in M$. The shape operator $S$ at $p$ is the linear transformation $S_p : T_pM \to T_pM$ given by

$$S_p(\vec{v}) = -\nabla_{\vec{v}} U \quad \text{for} \quad \vec{v} \in T_pM$$

where $U$ is the positively oriented, unit normal vector field on $M$.

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(b) Prove that if $\mathbf{v} \in T_pM$, then $S(\mathbf{v}) \in T_pM$.

We must show $S(\vec{v}) \cdot U(p) = 0$.

Well since $U$ has length 1:

$1 = U \cdot U$ So

$$0 = \vec{v}[U \cdot U] = (\nabla_{\vec{v}} U) \cdot U + U \cdot (\nabla_{\vec{v}} U) = 2(\nabla_{\vec{v}} U) \cdot U$$

So since $S(\vec{v}) = -\nabla_{\vec{v}} U$ we get

$$S(\vec{v}) \cdot U = 0.$$

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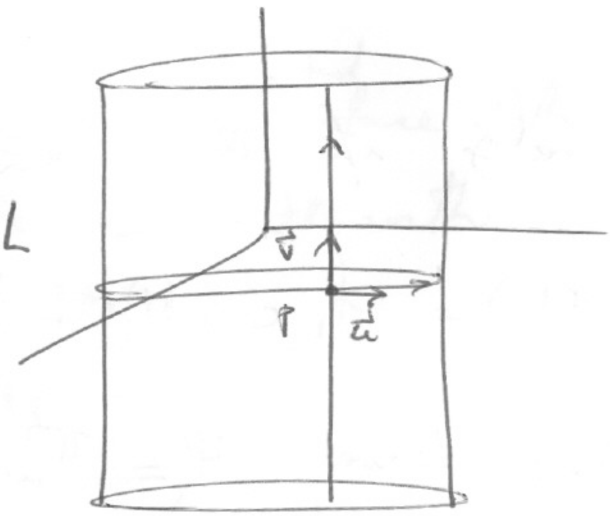
(c) Find the shape operator on the cylinder $x^2+y^2=r^2$ of radius $r$ with outward normal. Also find the principal curvatures and directions and use them to calculate the normal curvature at the point $(r,0,0)$ in the direction of the unit vector $(0, 1/\sqrt{2}, 1/\sqrt{2})$. 2 2

At each $p \in M$, let $\vec{u}$ be tangent vector to horizontal circle C through $p$ and $\vec{v}$ tangent vector to vertical line L thru $p$.




Since the normal is constant along L, $\boxed{S(\vec{v}) = \vec{0}}$.

If C is in plane $z = z_0$, then C has parametrization

$$\alpha(\theta) = (r\cos\theta, r\sin\theta, z_0)$$

and if $p = \vec{x}(\theta_0, z_0)$, where $\vec{x}(\theta, z) = (r\cos\theta, r\sin\theta, z)$

then
$$\vec{u} = \frac{\alpha'(\theta_0)}{\|\alpha'(\theta_0)\|} = (-\sin\theta_0, \cos\theta_0, 0) \qquad \alpha'(\theta_0) = r\vec{u}$$

Now
$$S(\vec{u}) = -\nabla_{\vec{u}} U = -\frac{1}{r}\nabla_{r\vec{u}} U = -\frac{1}{r}\nabla_{\alpha'(\theta_0)} U$$
$$= -\frac{1}{r}\frac{d}{d\theta}\Big|_{\theta=\theta_0} U(\alpha(\theta)) = -\frac{1}{r}\frac{d}{d\theta}\Big|_{\theta=\theta_0}(\cos\theta, \sin\theta, 0)$$

$$\boxed{S(\vec{u}) = -\frac{1}{r}\vec{u}.}$$

So matrix for $S$ in basis $\{\vec{u}, \vec{v}\}$ is $\begin{pmatrix} -1/r & 0 \\ 0 & 0 \end{pmatrix}$. PTO

So $k_1 = 0$, $k_2 = -\frac{1}{r}$

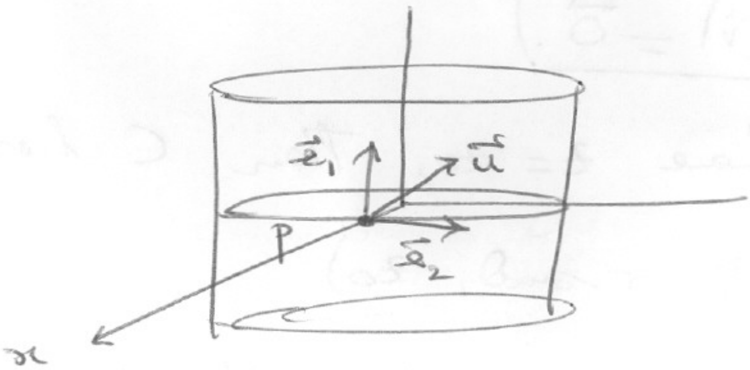
$\vec{e}_1 = \vec{v}$ $\vec{e}_2 = \vec{u}$.

Finally if $\vec{u} = \cos(\varphi)\,\vec{e}_1 + \sin(\varphi)\,\vec{e}_2$ then

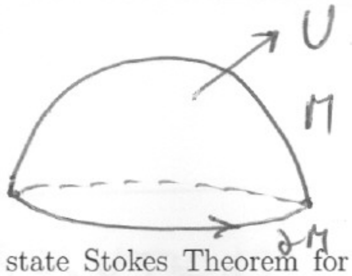
$k(\vec{u}) = k_1 \cos^2(\varphi) + k_2 \sin^2(\varphi)$

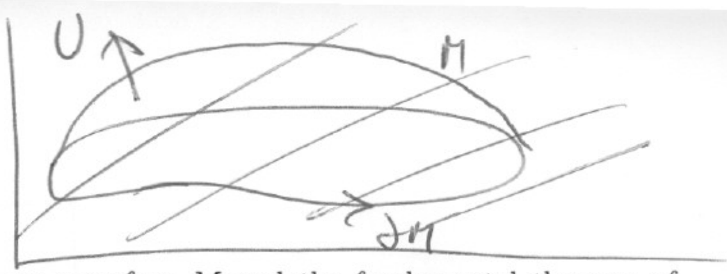
We have $\varphi = \pi/2$ for $\vec{u} = (0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$

at $(r, 0, 0)$




So $k(\vec{u}) = 0. + -\frac{1}{r}\sin^2(\pi/4) = -\frac{1}{2r}$




(5) [15 pts]

(a) Carefully state Stokes Theorem for vector fields on a surface $M$ and the fundamental theorem of calculus for differential forms on a surface $M$.

STOKES THM FOR VFS

Let V be a Vector Field on M, where M is an oriented surface with boundary $\partial M$ in $\mathbb{R}^3$. Orient the curve $\partial M$ with induced orientation as shown above.

Then $$\iint_M (\nabla \times V)\cdot d\vec{S} = \int_{\partial M} \vec{V}\cdot d\vec{s}.$$ 5

FTC Let $\omega$ be a 1-form on $M$, where $M$ is as above.

Then $$\iint_M d\omega = \int_{\partial M} \omega$$ 5

(c) Carefully and concisely explain how these two theorems are related.

THEY ARE THE SAME THEOREM STATED WITH DIFFERENT MATH. TERMINOLOGY

- 1-forms on M are in 1-1 correspondence with vector fields using the mapping $\omega_{\vec{V}}(\vec{W}) = \vec{V}\cdot\vec{W}$. 5
- 2-forms $\eta$ on M are in 1-1 correspondence with vector fields $\vec{X}$ on M using the mapping $\eta_{\vec{X}}(\vec{Y},\vec{Z}) = \vec{X}\cdot(\vec{Y}\times\vec{Z})$
- The d operator from 1-forms to 2-forms then corresponds to the curl operator on vector fields, i.e. $d\omega_{\vec{V}} = \eta_{\nabla\times\vec{V}}$
- Finally the def^ns of integrals give $\rightarrow \iint_M \eta_{\vec{X}} = \iint_M \vec{X}\cdot d\vec{S}$

  $\rightarrow \cancel{\iint} \int_{\partial M} \omega_{\vec{V}} = \int_{\partial M} \vec{V}\cdot d\vec{s}$.

Pledge: *I have neither given nor received aid on this exam*

Signature: ______________________________