MATH 423/673 EXAM 1 SOLUTIONS 2008.

1. $T(s) = \beta'(s) = \left(-\frac{1}{\sqrt{2}}\sin s,\ \cos s,\ -\frac{1}{\sqrt{2}}\sin s\right)$

$T'(s) = \left(-\frac{1}{\sqrt{2}}\cos s,\ -\sin s,\ -\frac{1}{\sqrt{2}}\cos s\right)$

$\kappa(s) = \|T'(s)\| = \underline{1}$

$N = T'/\kappa = T'$

$B = T \times N = \left(-\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right)$

$B' = (0,0,0) = -\tau N \Rightarrow \tau(s) \equiv 0.$

The curve is a circle radius $\underline{1}$ (as $\tau \equiv 0$ and $\kappa = 1$) with center $(0,0,0)$ in the plane $z = x$ (as normal to plane is B)

NB The reason center is $(0,0,0)$ is because
$\|\beta(s) - (0,0,0)\| = \|\beta(s)\| = 1$

2. (a) $T_pM = \{\vec{v} \in T_p\mathbb{R}^3 \,/\, \vec{v} = \alpha'(0)$ where $\alpha: \mathbb{R} \to M$ is a curve on M with $\alpha(0) = p\}$

(b) See Lemma 3.6 in section 4.3 and its proof.

3. $$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & yz & xz \end{vmatrix} = -y\vec{i} - z\vec{j} - x\vec{k}$$

$\nabla \times \vec{F}\,(1,2,3) = (-2,-3,-1)$

Circulation of $\vec{F}$ is greatest about the axis give by unit vector $\vec{v} = \dfrac{(-2,-3,-1)}{\sqrt{14}}$

4 a) The curve $\alpha(u) = (\sinh u, 0, \cosh u)$ in the xz plane is rotated about z axis to generate M.
$\vec{x}$ is of the form

$$\vec{x}(u,v) = (h(u)\cos v,\ h(u)\sin v,\ g(u))$$

with $h(u) = \sinh u$, $g(u) = \cosh u$.
This is ~~the~~ general form of a surface of revolution generated as explained above from $\alpha(u) = (h(u), 0, g(u))$

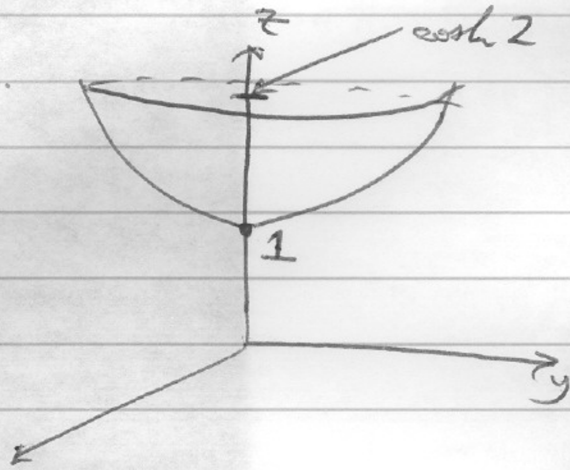
b) $x = \sinh u \cos v$, $y = \sinh u \sin v$, $z = \cosh u$

$x^2 + y^2 = \sinh^2 u$ $\quad z^2 = \cosh^2 u$

So ~~since~~ $\cosh^2 u - \sinh^2 u = 1$

$$z^2 - x^2 - y^2 = 1$$

So $F(x,y,z) = z^2 - x^2 - y^2 - 1$.



NOTE As $z = \cosh u > 0$ for all u we only have the top half of a hyperboloid of 2 sheets.
Since $0 < u < 2$, we only have the portion of the hyperboloid under plane $z = \cosh 2$.

c) $\alpha(t) = p + t\vec{v} = \left(2t, \frac{\pi}{3} + 3t\right)$

$\beta(t) = \vec{x}(\alpha(t))$

$$= \left(\sinh 2t \cos\left(\tfrac{\pi}{3} + 3t\right),\ \sinh 2t \sin\left(\tfrac{\pi}{3} + 3t\right),\ \cosh 2t\right)$$

$$\beta'(t) = \Big(2\cosh 2t \cos\left(\tfrac{\pi}{3} + 3t\right) - 3\sinh 2t \sin\left(\tfrac{\pi}{3} + 3t\right),$$
$$2\cosh 2t \sin\left(\tfrac{\pi}{3} + 3t\right) + 3\sinh 2t \cos\left(\tfrac{\pi}{3} + 3t\right),\ 2\sinh 2t\Big)$$

$$\vec{x}_*(\vec{v}_p) = \beta'(0) = \left(2\cos \pi/3,\ 2\sin \pi/3,\ 0\right)$$
$$= (1, \sqrt{3}, 0)$$

(d) $\displaystyle\iint_M \vec{F}\cdot d\vec{A} = \iint_D \vec{F}(\vec{x}(u,v))\cdot(\vec{x}_u \times \vec{x}_v)\,du\,dv$

where D is $0 < u < 2, \quad 0 < v < 2\pi$

$F(\vec{x}(u,v)) = (\cosh u, \cosh u, \sinh^2 u)$

$\vec{x}_u = (\cosh u \cos v, \cosh u \sin v, \sinh u)$
$\vec{x}_v = (-\sinh u \sin v, \sinh u \cos v, 0)$

$$F(\vec{x}(u,v))\cdot \vec{x}_u \times \vec{x}_v = \begin{vmatrix} \cosh u & \cosh u & \sinh^2 u \\ \cosh u \cos v & \cosh u \sin v & \sinh u \\ -\sinh u \sin v & \sinh u \cos v & 0 \end{vmatrix}$$

$-\cosh u \sinh^2 u \cos v$

$= \cancel{\cosh^2 u \sin v} - \cosh u \sinh^2 u \sin v$
$\quad + \sinh^2 u(\cosh u \sinh u \cos^2 v + \cosh u \sinh u \sin^2 v)$
$= -\cosh u \sinh^2 u(\cos v + \sin v) + \cosh u \sinh^3 u.$

So

$$\iint_M \vec{F}\cdot d\vec{A} = \int_0^{2\pi}\int_0^2 \cosh u \sinh^3 u - \cosh u \sinh^2 u(\cos v + \sin v)\,du\,dv$$

Since $\int_0^{2\pi} \cos v\,dv = \int_0^{2\pi} \sin v\,dv = 0$ we have

$$\iint_M \vec{F}\cdot d\vec{A} = 2\pi \int_0^2 \cosh u \sinh^3 u\,du$$

$w = \sinh u$
$dw = \cosh u\,du$

$$= 2\pi \int_0^{\sinh 2} w^3\,dw$$
$$= 2\pi \left.\frac{w^4}{4}\right]_0^{\sinh 2} = \underline{\frac{\pi}{2}\sinh^4 2}$$

5 $$L(\varepsilon) = \int_a^b \|\beta_\varepsilon'(s)\|\, ds$$

$T = \alpha'$

$N' = -\kappa T + \tau B$

$$= \int_a^b \|\alpha'(s) + \varepsilon N'(s)\|\, ds$$

$$= \int_a^b \| T + \varepsilon(-\kappa T + \tau B)\|\, ds$$

$$= \int_a^b \|(1-\varepsilon\kappa) T + \varepsilon\tau B\|\, ds$$

$$= \int_a^b \sqrt{[(1-\varepsilon\kappa)T + \varepsilon\tau B]\cdot[(1-\varepsilon\kappa)T + \varepsilon\tau B]}\, ds$$

$$= \int_a^b \sqrt{(1-\varepsilon\kappa)^2 + \varepsilon^2\tau^2}\, ds$$ as $T\cdot T = 1 = B\cdot B$ and $T\cdot B = 0$

$$\boxed{L(\varepsilon) = \int_a^b \sqrt{(1-\varepsilon\kappa)^2 + \varepsilon^2\tau^2}\, ds}$$

$$L'(\varepsilon) = \frac{d}{d\varepsilon}\int_a^b \sqrt{(1-\varepsilon\kappa)^2 + \varepsilon^2\tau^2}\, ds$$

$$= \int_a^b \frac{d}{d\varepsilon}\left(\sqrt{(1-\varepsilon\kappa)^2 + \varepsilon^2\tau^2}\right) ds$$

$$L'(\varepsilon) = \int_a^b \tfrac{1}{2}\left((1-\varepsilon\kappa)^2 + \varepsilon^2\tau^2\right)^{-\frac{1}{2}} \cdot \left[2(1-\varepsilon\kappa)(-\kappa) + 2\varepsilon\tau^2\right] ds$$

$$L'(0) = \int_a^b \tfrac{1}{2}(1)^{-1/2}(-2\kappa)\, ds$$

$$= -\int_a^b \kappa(s)\, ds$$ as required.

If α is not a straight line then $\kappa > 0$ holds at least on some subinterval of $[a,b]$, so $L'(0) < 0$ holds. So length of β_ε is decreasing as ε increases at $\varepsilon = 0$.