NAME: SOLUTIONS

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MATH 423/673 (Spring 2008) Exam 2, April 21st

No calculators, books or notes!
Show all work and give **complete explanations** for all your answers.
This is a 75 minute exam. It is worth a total of 75 points.

(1) [20 pts]
(a) Let ω be the 1-form on $\mathbf{R}^2$ defined by $\omega = y\,dx + x^2 dy$. Let p be the point $p = (2, 4)$ and $\mathbf{v}_p \in T_p\mathbf{R}^2$ the vector $\mathbf{v}_p = (3, 7)$. Calculate $\omega(\mathbf{v}_p)$.

$$\omega_p(\vec{v}_p) = 4 \times 3 + 2^2 \times 7$$

$$= 12 + 28 = 40$$

$dx(\vec{v}_p) = 3$
$dy(\vec{v}_p) = 7$
$y(p) = 4$
$x(p) = 2$

(b) Let $F : \mathbf{R}^2 \to \mathbf{R}^2$ be the mapping $F(x, y) = (xy, x^2 - y^2)$ and let ω, p and $\mathbf{v}_p$ be as in (a) above. Calculate $(F^*\omega)(\mathbf{v}_p)$.

$$(F^*\omega)_p(\vec{v}_p) = \omega_{F(p)}(F_*(\vec{v}_p))$$

NOW if we write $F(x,y) = (f_1(x,y), f_2(x,y)) = (xy, x^2 - y^2)$

Then $F_*(\vec{v}_p) = (\vec{v}_p[f_1], \vec{v}_p[f_2]) = (\nabla f_1(p) \cdot \vec{v}_p, \nabla f_2(p) \cdot \vec{v}_p)$

Now $\nabla f_1 = y\vec{i} + x\vec{j}$, $\quad \nabla f_1(p) \cdot \vec{v}_p = 4 \times 3 + 2 \times 7 = 26$

$\nabla f_2 = 2x\vec{i} - 2y\vec{j}$, $\quad \nabla f_2(p) \cdot \vec{v}_p = 2.2.3 - 2.4.7 = -44$

So $F_*(\vec{v}_p) = (26, -44)$ is a vector at $F(p) = (2.4, 2^2 - 4^2) = (8, -12)$

So

$$(F^*\omega)_p(\vec{v}_p) = \omega_{F(p)}(F_*(\vec{v}_p)) = -12.26 + 64(-44) = -312 - 2816$$

(c) Let η be the 1-form on $\mathbf{R}^3$ defined by $\eta = xydx + ye^z dy + \sin(x)y^2 dz$. Calculate $d\eta$.

$$d\eta = d(xy)\wedge dx + d(ye^z)\wedge dy + d(y^2\sin x)\wedge d$$

$$= (y\,dx + x\,dy)\wedge dx + (e^z dy + ye^z dz)\wedge dy$$

$$+ 2y\sin x\, dy\wedge dz + y^2\cos x\, dx\wedge dz$$

$$= x\,dy\wedge dx + ye^z\, dz\wedge dy + 2y\sin x\, dy\wedge dz$$

$$+ y^2\cos x\, dx\wedge dz$$

Here I u
$dx\wedge dx$
$dx\wedge dy =$
etc

$$= -x\,dx\wedge dy + y^2\cos x\, dx\wedge dz$$

$$+ (2y\sin x - ye^z)\,dy\wedge dz$$

(d) Let ν be a 2-form on $\mathbf{R}^3$. Prove that for any tangent vectors $\mathbf{v}$ and $\mathbf{w}$ to $\mathbf{R}^3$ at a point p that

$$\nu(a\mathbf{v} + b\mathbf{w}, c\mathbf{v} + d\mathbf{w}) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \nu(\mathbf{v}, \mathbf{w}).$$

$$\nu(a\vec{v} + b\vec{w},\ c\vec{v} + d\vec{w})$$

$$= ac\,\nu(\vec{v},\vec{v}) + ad\,\nu(\vec{v},\vec{w}) + bc\,\nu(\vec{w},\vec{v}) + bd\,\nu(\vec{w}$$

by linearity

$$= (ad - bc)\,\nu(\vec{v},\vec{w}) \quad \text{as} \quad \nu(\vec{v},\vec{v}) = \nu(\vec{w},\vec{w}) = 0$$

and $\nu(\vec{v},\vec{w}) = -\nu(\vec{w},\vec{v})$

$$= \begin{vmatrix} a & b \\ c & d \end{vmatrix} \nu(\vec{v},\vec{w}).$$

(2) [20 pts]
(a) Calculate the Gauss curvature function on the elliptic paraboloid $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$.

Use patch $\vec{x}(u,v) = \left(u, v, \left(\frac{u}{a}\right)^2 + \left(\frac{v}{b}\right)^2\right)$

$\vec{x}_u = \left(1, 0, \frac{2u}{a^2}\right) \qquad \vec{x}_v = \left(0, 1, \frac{2v}{b^2}\right)$

$E = \vec{x}_u \cdot \vec{x}_u = 1 + \left(\frac{2u}{a^2}\right)^2, \quad G = \vec{x}_v \cdot \vec{x}_v = 1 + \left(\frac{2v}{b^2}\right.$

$F = \vec{x}_u \cdot \vec{x}_v = \frac{4uv}{a^2 b^2}$

$$\vec{x}_u \times \vec{x}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & \frac{2u}{a^2} \\ 0 & 1 & \frac{2v}{b^2} \end{vmatrix} = \left(-\frac{2u}{a^2}, -\frac{2v}{b^2}, 1\right)$$

$\|\vec{x}_u \times \vec{x}_v\|^2 = 1 + \left(\frac{2u}{a^2}\right)^2 + \left(\frac{2v}{b^2}\right)^2$

$$U = \frac{\vec{x}_u \times \vec{x}_v}{\|\vec{x}_u \times \vec{x}_v\|} = \frac{1}{\sqrt{1 + \left(\frac{2u}{a^2}\right)^2 + \left(\frac{2v}{b^2}\right)^2}}\left(-\frac{2u}{a^2}, -\frac{2v}{b^2}, 1\right.$$

$\vec{x}_{uu} = \left(0, 0, \frac{2}{a^2}\right), \quad \vec{x}_{vv} = \left(0, 0, \frac{2}{b^2}\right), \quad \vec{x}_{uv} = (0, 0,$

So $L = U \cdot \vec{x}_{uu} = \dfrac{2/a^2}{\sqrt{1 + \left(\frac{2u}{a^2}\right)^2 + \left(\frac{2v}{b^2}\right)^2}}$

$M = U \cdot \vec{x}_{uv} = 0$

$N = U \cdot \vec{x}_{vv} = \dfrac{2/b^2}{}$

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$$K = \frac{LN - M^2}{EG - F^2}$$

$$= \frac{LN}{EG - F^2} = \frac{LN}{\|\vec{x}_u \times \vec{x}_v\|^2}$$

$$= \frac{\dfrac{4}{a^2 b^2}}{1 + \left(\dfrac{2u}{a^2}\right)^2 + \left(\dfrac{2v}{b^2}\right)^2} \quad \frac{1}{1 + \left(\dfrac{2u}{a^2}\right)^2 + \left(\dfrac{2v}{b^2}\right)^2}$$

$$= \frac{4/(ab)^2}{\left[1 + \left(\dfrac{2u}{a^2}\right)^2 + \left(\dfrac{2v}{b^2}\right)^2\right]^2}.$$

(b) What is the maximum value of the Gauss curvature on this surface and

$$K = \frac{4/a^2b^2}{\left(1+\left(\frac{2u}{a^2}\right)^2+\left(\frac{2v}{b^2}\right)^2\right)^2}$$ is MAX

is at $u = v = 0$. where

$$\boxed{K(0,0) = \frac{4}{a^2b^2}}$$

This ~~is~~ at $(0,0,0)$ on M

(3) [20 pts]
(a) Let Z be a Euclidean vector field on a surface M in $\mathbf{R}^3$ and let $\mathbf{v} \in T_pM$
a point p. State the definition of the covariant derivative $\nabla_{\mathbf{v}} Z$.

Let $\alpha : \mathbb{R} \to M \subset \mathbb{R}^3$ be a curve

$\alpha(0) = p$ and $\alpha'(0) = \vec{v}$.

Let $Z_\alpha(t) := Z(\alpha(t))$ ~~be~~

(So $Z_\alpha : \mathbb{R} \to \mathbb{R}^3$ holds.)

Then

$$\nabla_{\vec{v}} Z := Z_\alpha'(0) \in T_p\mathbb{R}^3$$

FROM BOTTOM: RHS of ⊛ $= \dfrac{-uv}{(1+u^2+v^2)^{3/2}}\vec{x}_u + \dfrac{1+v^2}{(1+u^2+v^2)^{3/2}}\vec{x}_v$

$$= \frac{1}{(1+u^2+v^2)^{3/2}}\left(-uv,\ 1+v^2,\ -uv^2+u+uv^2\right) = S(\vec{x}_u) \text{ by } ⊛⊛$$

(b) Let S be the shape operator on the saddle surface M with patch $\mathbf{x}(u,v) = (u, v, uv)$. Show that

$$S(\mathbf{x}_u) = \frac{-uv}{(1+u^2+v^2)^{3/2}}\mathbf{x}_u + \frac{1+v^2}{(1+u^2+v^2)^{3/2}}\mathbf{x}_v. \qquad ⊛$$

[Hint: You may use the fact that if Z is a Euclidean vector field on M and $\overline{Z}(u,v) = Z(\mathbf{x}(u,v))$ is its coordinate expression, then the coordinate expression for $\nabla_{\mathbf{x}_u} Z$ is $\frac{\partial \overline{Z}}{\partial u}$.]

The right hand side of ⊛ is really the coordinate expression of the tangent vector $S(\vec{x}_u)$ in the coordinate patch $\vec{x} = \vec{x}(u,v)$.

We have $U = \dfrac{\vec{x}_u \times \vec{x}_v}{\|\vec{x}_u \times \vec{x}_v\|}$

$\vec{x}_u = (1, 0, v)$, $\vec{x}_v = (0, 1, u)$

So $\vec{x}_u \times \vec{x}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & v \\ 0 & 1 & u \end{vmatrix} = (-v, -u, 1)$

$$U = \frac{(-v, -u, 1)}{\sqrt{1+u^2+v^2}} = (U_1, U_2, U_3)$$

So $S(\vec{x}_u) = -\nabla_{\vec{x}_u} U = -\dfrac{\partial U}{\partial u} = -\left(\dfrac{\partial U_1}{\partial u}, \dfrac{\partial U_2}{\partial u}, \dfrac{\partial U_3}{\partial u}\right)$

$\dfrac{\partial U_1}{\partial u} = -v\dfrac{\partial}{\partial u}(1+u^2+v^2)^{-1/2} = \dfrac{uv}{(1+u^2+v^2)^{3/2}}$ etc gives

$S(\vec{x}_u) = \dfrac{(-uv,\ 1+v^2,\ u)}{}$ ⊛⊛

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(c) Use the result in (b) together with the analogous formula

$$S(\mathbf{x}_v) = \frac{1+u^2}{(1+u^2+v^2)^{3/2}}\mathbf{x}_u - \frac{uv}{(1+u^2+v^2)^{3/2}}\mathbf{x}_v.$$

to calculate the normal curvature, $k(\mathbf{u})$, of the saddle surface for an arbitrary unit tangent vect surface at the point $p = (1, 0, 0)$. [Hint: Use the fact that $\mathbf{x}_u$ and $\mathbf{x}_v$ are orthogonal at p to ex terms of $\mathbf{x}_u$ and $\mathbf{x}_v$.]

AT $p = (1,0,0)$ $\quad \boxed{u = 1, \quad v = 0}$

$\vec{x}_u = (1,0,0) = \vec{i}, \qquad \vec{x}_v = (0,1,1) = \vec{j} + \vec{k}.$

$\|\vec{x}_u\| = 1, \qquad \|\vec{x}_v\| = \sqrt{2}, \qquad \vec{x}_u \cdot \vec{x}_v = 0.$

So we can write our unit vector $\vec{u}$ as

$$\boxed{\vec{u} = \vec{u}(\theta) = \cos\theta\, \vec{x}_u + \sin\theta \frac{\vec{x}_v}{\sqrt{2}}}$$ a

Now at p

$S(\vec{x}_u) = \frac{1}{2^{3/2}}\vec{x}_v = 2^{-3/2}\vec{x}_v$

$S(\vec{x}_v) = \frac{2}{2^{3/2}}\vec{x}_u = \frac{1}{\sqrt{2}}\vec{x}_u$

$S(\vec{x}_u)\cdot\vec{x}_u = 0, \qquad S(\vec{x}_v)\cdot\vec{x}_v = 0, \qquad$ ~~$S(\vec{x}_u)\cdot\vec{x}_v = \frac{1}{\ldots}$~~

$S(\vec{x}_u)\cdot\vec{x}_v = \frac{1}{2^{3/2}}\vec{x}_v\cdot\vec{x}_v = \frac{1}{2^{3/2}}\cdot 2 = \frac{1}{\sqrt{2}} = S($

So

$\underline{\underline{k(\vec{u}) = k(\theta)}} = S(\vec{u})\cdot\vec{u} = [\cos\theta\, S(\vec{x}_u) + \frac{1}{\sqrt{2}}\sin\theta\, S(\vec{x}_v$

$= 2\cos\theta\sin\theta\, S(\vec{x}_u)\cdot\vec{x}_v = \frac{1}{\ldots}\sin 2\theta$

(d) Use your answer to (c) to find the principal curvatures and principal directions to M at $p = (1, 0, 0)$.

$k(\theta) = \frac{1}{\sqrt{2}} \sin 2\theta$ is maximum at $\theta = \pi/4, 5\pi/4$

So $k_1 = k_{MAX} = k(\pi/4) = \cancel{\ } \frac{1}{2}$

and $\vec{v}_1 = \vec{u}(\pi/4) = \cos(\pi/4)\vec{x}_u + \frac{1}{\sqrt{2}}\sin(\pi/4)\vec{x}_v$

$= \frac{1}{\sqrt{2}}(1, 0, 0) + \left(\frac{1}{\sqrt{2}}\right)^2 (0, 1, 1) = \cancel{\frac{1}{\sqrt{2}}(1,1,1}$

$= \left(\frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{2}\right)$

(or $\vec{v}_1 = \vec{u}(5\pi/4) = -\frac{1}{\sqrt{2}}(1, 1, 1)$

<u>AND</u> $k_2 = k_{MIN} = k(3\pi/4) = \cancel{\ } -\frac{1}{2}$ and

$\vec{v}_2 = \vec{u}(3\pi/4) = \cos(3\pi/4)\vec{x}_u + \sin(3\pi/4)\vec{x}_v = \cancel{\frac{1}{\sqrt{2}}(-1,1,1)}$

$= \left(-\frac{1}{\sqrt{2}}, \frac{1}{2}, \ldots\right)$

(4) [15 pts] Carefully show why Stokes' Theorem is a special case of the Fundamental Theorem of Calculus for Differential Forms. [*There is more space on the next page.*]

The FTC for Differential Forms says that if ω is a p-form and M is a $(p+1)$-dimensional manifold with boundary ∂M of dimension $(p-1)$, Then

$$\int_M d\omega = \int_{\partial M} \omega \qquad (1)$$

where $d\omega$ is the $(p+1)$-form obtained by taking the exterior derivative of ω.

Stokes' Theorem says that if $\vec{F}$ is a vectorfield on $\mathbb{R}^3$ and $M \subset \mathbb{R}^3$ is a 2D surface, Then

$$\iint_M (\nabla \times \vec{F}) \cdot d\vec{A} = \int_{\partial M} \vec{F} \cdot d\vec{A}. \qquad (2)$$

~~Provided~~ Here we must choose an orientation on M

∂M with the induced orientation you get using the right-hand-rule.

Let's show that when $p = 1$ that ① becomes ②

provided M is a subset of $\mathbb{R}^3$.

First recall that ~~vector fields~~ 1-forms ω on $\mathbb{R}^3$ are in 1-1 correspondence with vector fields $\vec{F}$ on $\mathbb{R}^3$. This correspondence is given by

$$\omega = f\,dx + g\,dy + h\,dz \iff \vec{F} = f\vec{i} + g\vec{j} + h\vec{k}.$$

By the definition of $\int_{\partial M} \omega$ we have

$$\int_{\partial M} \omega = \int_a^b \omega_{\alpha(t)}(\alpha'(t))\,dt = \int_a^b \Big[f(\alpha(t))\frac{dx}{dt} + g(\alpha(t))\frac{dy}{dt} + h(\alpha(t))\frac{dz}{dt}\Big]dt$$

where $\alpha : [a, b] \longrightarrow \mathbb{R}^3$ is a parametrization of ∂M.

But $$\int_{\partial M} \vec{F}\cdot d\vec{r} = \int_a^b \Big[f(\alpha(t))\frac{dx}{dt} + g(\alpha(t))\frac{dy}{dt} + h(\alpha(t))\frac{dz}{dt}\Big]dt$$

too. Here $\alpha(t) = (x(t), y(t), z(t))$.

So $$\int_{\partial M} \omega = \int_{\partial M} \vec{F}\cdot d\vec{r}.$$

Second there is a 1-1 correspondence between 2-forms η on $\mathbb{R}^3$ and VFs $\vec{G}$ on $\mathbb{R}^3$, given by

$$\eta(\vec{v}, \vec{w}) = \vec{G}\cdot(\vec{v}\times\vec{w}) \quad (*)$$

Pledge: *I have neither given nor received aid on this exam*

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Signature:

$\underline{\text{or}}$ $\vec{G} = (h_1, h_2, h_3)$ corresponds to

$\eta = h_1\, dx_2 \wedge dx_3 + h_2\, dx_3 \wedge dx_1 + h_3\, dx_1 \wedge dx_2$

<u>Also</u>
$$\iint_M \eta = \iint_M \vec{G} \cdot d\vec{A}$$
under this correspondence. as if $\vec{x} : D \to M$ is a patch, then

$$\iint_M \eta = \iint_D \vec{x}^* \eta = \iint_D \eta(\vec{x}_u, \vec{x}_v)\, du\, dv$$
$$\underset{=}{\pm} \iint_D \vec{G} \cdot (\vec{x}_u \times \vec{x}_v)\, du\, dv$$
$$= \iint_M \vec{G} \cdot d\vec{A} \quad \text{by def}^n.$$

Finally if ω (a 1-form) corresponds to $\vec{F}$

Then $\eta = d\omega$ corresponds to $\vec{G} = \nabla \times \vec{F}$

since $\omega = f_1\, dx_1 + f_2\, dx_2 + f_3\, dx_3$, $\vec{F} = (f_1, f_2, f_3)$

$\eta = d\omega = \left(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}\right) dx_2 \wedge dx_3 + \cdots$ and

$\nabla \times \vec{F} = \left(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}\right) \vec{i} + \cdots$

So putting all this together