NAME: SOLUTIONS

1 /6	2 /12	3 /20	4 /9	5 /16	6 /12	T /75

MATH 423/673 (Spring 2010) Exam 1, March 9th

No calculators, books or notes!
Show all work and give **complete explanations** for all your answers.

(1) [6 pts] Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be the mapping defined by

$$F(x, y) = (ax, by) \qquad \text{for some constants } a, b > 0.$$

Show that the image under F of a circle centered at the origin is an ellipse centered at the origin.

A circle of radius R center $(0,0)$ is parametrized by

$$\alpha(t) = (R\cos t, R\sin t) \qquad 0 < t < 2\pi$$

The image under F of α is

$$\beta(t) = F(\alpha(t)) = (aR\cos t, bR\sin t)$$

If $x = aR\cos t$, $y = bR\sin t$ then

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = R^2(\cos^2 t + \sin^2 t) = R^2$$

which is the equation of an ellipse centered at $(0,0)$.

(2) [12 pts]
(a) Let C be the curve with parametrization $\alpha(t) = (\sin t, \cos t, t^2)$. Find a parametrization of the tangent line to α at $t = \frac{\pi}{4}$.

$$\alpha\left(\frac{\pi}{4}\right) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{\pi^2}{16}\right)$$

$$\alpha'(t) = (\cos t, -\sin t, 2t)$$

$$\alpha'(\pi/4) = (1/\sqrt{2}, -1/\sqrt{2}, \pi/2)$$

$$\vec{l}(s) = \alpha(\pi/4) + (s - \pi/4)\,\alpha'(\pi/4)$$

$$= \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{\pi^2}{16}\right) + \left(s - \frac{\pi}{4}\right)\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \pi/2\right)$$

(b) Let $\mathbf{v}_p$ be the tangent vector to $\mathbb{R}^3$ with $p = (3, -1, \frac{\pi}{2})$ and $v = (1, 4, \pi)$ and let $f : \mathbb{R}^3 \to \mathbb{R}$ be the function $f = xy + \cos z$. Working directly from the definition, compute the direction derivative $\mathbf{v}_p[f]$.

Let $\alpha(t) = \vec{p} + t\vec{v} = \left(3+t, -1+4t, \frac{\pi}{2} + t\pi\right)$

Then

$$\vec{v}_p[f] = \frac{d}{dt}\Big|_{t=0} f(\alpha(t))$$

$$= \frac{d}{dt}\Big|_{t=0} \left[(3+t)(-1+4t) + \cos\left(\frac{\pi}{2} + t\pi\right)\right]$$

$$= \left[1(-1+4t) + (3+t)4 - \pi \sin\left(\frac{\pi}{2} + t\pi\right)\right]\Big|_{t=0}$$

$$= -1 + 12 - \pi = 11 - \pi$$

(3) [20 pts] Consider the unit speed curve $\beta(s) = (1+\cos(s/2), \sqrt{3}\cos(s/2), 2\sin(s/2))$. Calculate the curvature, torsion, and Frenet frame, T, N, and B, of the curve β. Use your result to explain why the curve lies in a plane.

$$\beta(s) = \left(1+\cos\left(\tfrac{s}{2}\right),\ \sqrt{3}\cos\left(\tfrac{s}{2}\right),\ 2\sin\tfrac{s}{2}\right)$$

$$T(s) = \beta'(s) = \left(-\tfrac{1}{2}\sin\left(\tfrac{s}{2}\right),\ -\tfrac{\sqrt{3}}{2}\sin\left(\tfrac{s}{2}\right),\ \cos\tfrac{s}{2}\right)$$

<u>NOTE</u> $$\|T\|^2 = \left(\tfrac{1}{4}+\tfrac{3}{4}\right)\sin^2(s/2) + \cos^2(s/2) = 1$$

$$T'(s) = \left(-\tfrac{1}{4}\cos\left(\tfrac{s}{2}\right),\ -\tfrac{\sqrt{3}}{4}\cos(s/2),\ \tfrac{-1}{2}\sin\tfrac{s}{2}\right)$$

$$K(s) = \|T'(s)\| = \sqrt{\left(\tfrac{1}{16}+\tfrac{3}{16}\right)\cos^2(s/2) + \tfrac{1}{4}\sin^2(s/2)}$$

$$= \tfrac{1}{2}$$

$$N = \frac{T'}{\|T'\|} = \left(-\tfrac{1}{2}\cos(s/2),\ -\tfrac{\sqrt{3}}{2}\cos(s/2),\ -\sin s/2\right)$$

$$B = T\times N = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\tfrac{1}{2}\sin s/2 & -\tfrac{\sqrt{3}}{2}\sin s/2 & \cos s/2 \\ -1/2\cos s/2 & -\sqrt{3}/2\cos s/2 & -\sin s/2 \end{vmatrix}$$

$$= \left(\tfrac{\sqrt{3}}{2},\ -\tfrac{1}{2},\ 0\right) \text{ is indept of } s.$$

$$B' = -\tau N \Rightarrow \tau = 0 \text{ as } B' = \vec{0}.$$

Any curve with constant curvature and zero torsion
is a circle that lies in a plane.

(4) [9 pts]
(a) Let V be a vector field on $\mathbb{R}^3$, let $U_1 = (1,0,0)$, $U_2 = (0,1,0)$, $U_3 = (0,0,1)$ be the natural frame field on $\mathbb{R}^3$, and let $x_j : \mathbb{R}^3 \to \mathbb{R}$ be the j-th coordinate function defined by $x_j(p_1, p_2, p_3) = p_j$, for $j = 1, 2, 3$. Prove that

$$V = \sum_{j=1}^{3} V[x_j] U_j.$$

Since $\{U_1, U_2, U_3\}$ form a basis for $T_p\mathbb{R}^3$ we know

$$V = \sum_{j=1}^{3} \lambda_j U_j \quad \text{for some } \lambda_j = \lambda_j(p).$$

Now $$V[x_i] = \left(\sum_{j=1}^{3} \lambda_j U_j\right)[x_i] = \sum_{j=1}^{3} \lambda_j U_j[x_i]$$

$$\overset{*}{=} \sum_{j=1}^{3} \lambda_j \delta_{ij} = \lambda_i \qquad \text{So } \lambda_i = V[x_i]$$

So $V = \sum V[x_j] U_j$

$*$ $U_j[f] = \frac{\partial f}{\partial x_j}$ So $U_j[x_i] = \frac{\partial x_i}{\partial x_j} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

$$\boxed{U_j[f] = \frac{d}{dt}\Big|_{t=0} f(\vec{p} + tU_j) = \frac{\partial f}{\partial x_j}(\vec{p})}$$

(b) Suppose that V and W are vector fields on $\mathbb{R}^3$ so that

$$V[f] = W[f] \qquad \text{for all functions } f : \mathbb{R}^3 \to \mathbb{R}.$$

Prove that $V = W$.

From (a) we know

$$V = \sum V[x_j] U_j, \quad W = \sum W[x_j] U_j$$

~~So~~ and choosing $f = x_j$ $(j = 1, 2, 3)$ gives

$$\lambda_j = V[x_j] = W[x_j] \quad \text{from our assumption}$$

So $$V = \sum \lambda_i U_i = W.$$

(5) [16 pts] Let $\mathbf{x} : D \to \mathbb{R}^3$ be the mapping defined by

$$\mathbf{x}(u,v) = (\sinh u \cos v, \frac{1}{2}\sinh u \sin v, \cosh u),$$

where D is the domain in $\mathbb{R}^2$ given by $0 < u < \infty$ and $0 < v < 2\pi$. [Recall that $\cosh u = \frac{1}{2}(e^u + e^{-u})$, $\sinh u = \frac{1}{2}(e^u - e^{-u})$ and $\cosh^2 u - \sinh^2 u = 1$.]

(a) Let $M = \mathbf{x}(D)$ be the image of D. Find an equation of the form $F(x,y,z) = 0$ that is satisfied by all points on M.

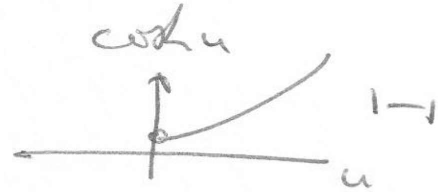
$$z^2 - x^2 - 4y^2 = \cosh^2 u - \sinh^2 u\left[\cos^2 v + 4\left(\tfrac{1}{2}\sin v\right)^2\right]$$
$$= \cosh^2 u - \sinh^2 u = 1$$

So $F(x,y,z) = z^2 - x^2 - 4y^2 - 1$

(b) Prove that $\mathbf{x}$ is 1-1 on D. What goes wrong when $u = 0$?

Suppose $\vec{x}(u_1, v_1) = \vec{x}(u_2, v_2)$ where $u_1, u_2 > 0$, $v_1, v_2 \in (0, 2\pi)$

So using z coordinate of $\vec{x}$:



$\cosh u_2 = \cosh u_1$

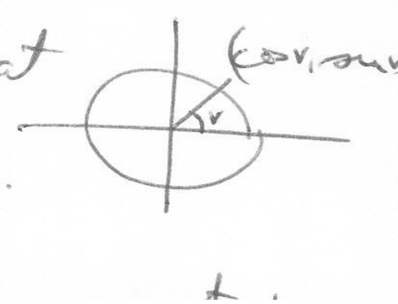
$\Rightarrow u_2 = u_1$ as $\cosh$ is 1-1 on $(0, \infty)$

$\frac{d}{dx}\cosh x = \sinh x > 0$ for $x > 0$ So $\cosh x \uparrow$ So $\cosh x$ 1-1

Then using x, y coords of $\vec{x}$:

$(\cos v_1, \sin v_1) = (\cos v_2, \sin v_2)$

So since each point on circle has at most one angle v in $(0, 2\pi)$, $v_1 = v_2$.



$\boxed{\text{IF } u=0}$ $\vec{x}(0,v) = (0, 0, 1)$ ...

(c) Prove that $\mathbf{x}$ is regular on D. What goes wrong when $u = 0$?

$$D\vec{x} = \begin{bmatrix} \frac{\partial \vec{x}}{\partial u} & \frac{\partial \vec{x}}{\partial v} \end{bmatrix} = \begin{bmatrix} \cosh u \cos v & -\sinh u \sin v \\ \frac{1}{2}\cosh u \sin v & \frac{1}{2}\sinh u \cos v \\ \sinh u & 0 \end{bmatrix}$$

If $u > 0$ then $\sinh u > 0$, so $\frac{\partial \vec{x}}{\partial u} \neq \vec{0}$ and $\frac{\partial \vec{x}}{\partial u} \neq \lambda \frac{\partial \vec{x}}{\partial v}$. We just need to check $\frac{\partial \vec{x}}{\partial v} \neq \vec{0}$.

Well $\left\| \frac{\partial \vec{x}}{\partial v} \right\|^2 = \sinh^2 u \left[\sin^2 v + \frac{1}{4}\cos^2 v \right] > 0$

as $\sinh u > 0$ and $\cos v$, $\sin v$ cannot be simultaneously 0. So $Rk(D\vec{x}) = 2$ on D.

So $\vec{x}$ is regular on D.

$\boxed{\text{When } u = 0}$ $D\vec{x} = \begin{bmatrix} \cos v & 0 \\ \frac{1}{2}\sin v & 0 \\ 0 & 0 \end{bmatrix}$ which has rank 1.

So $\vec{x}$ not regular for $u=0$

(6) [12 pts]

(a) Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a mapping. Define the tangent mapping $F_* : T_p\mathbb{R}^n \to T_{F(p)}\mathbb{R}^m$.

Let ~~$\alpha(t)$~~ $\vec{v} \in T_p \mathbb{R}^n$ and $\alpha(t) = \vec{p} + t\vec{v}$ be the line thru $\vec{p}$ with tangent $\vec{v}$.

Define $\beta(t) = F(\alpha(t)) = F(\vec{p} + t\vec{v})$.

$\beta(0) = F(\vec{p})$.

Set $F_*(\vec{v}_p) := \beta'(0) \in T_{F(p)} \mathbb{R}^m$.

Starting with the
(b) ~~Using the~~ definition in (a) prove the following version of the Chain Rule: If $F : \mathbb{R}^n \to \mathbb{R}^m$ and $G : \mathbb{R}^m \to \mathbb{R}^k$ are mappings then

$$(G \circ F)_* = G_* \circ F_*.$$

$$(G\circ F)_*(\vec{v}_p) = \frac{d}{dt}\Big|_{t=0} (G\circ F)(\vec{p}+t\vec{v}) = \frac{d}{dt}\Big|_{t=0} G(F(\vec{p}+t\vec{v}))$$

$$= \frac{d}{dt}\Big|_{t=0} G(\beta(t)) \text{ where } \beta(0) = F(\vec{p}), \quad \beta'(0) = F_*(\vec{v}).$$

$$\overset{\circledast}{=} G_*(F_*(\vec{v}_p))$$

$$= (G_* \circ F_*)(\vec{v}_p)$$

$\circledast$ PTO

(c) Hence show that if $F : \mathbb{R}^n \to \mathbb{R}^n$ is invertible then F_* is invertible and

$$(F_*)^{-1} = (F^{-1})_*.$$

Suppose F is invertable. Then

$$(F\circ F^{-1}) = I = (F^{-1}\circ F)$$

~~So by (b)~~ $(F\circ F^{-1})_* = I_* = (F^{-1}\circ F)_*$

So by (b), $F_* \circ (F^{-1})_* = I = (F^{-1})_* F_*$ ← So F_* is invertible

ie $(F^{-1})_* = (F_*)^{-1}$.

Pledge: *I have neither given nor received aid on this exam*

Signature: ______________________

LEMMA

Let $F : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and α any curve with $\alpha(0) = p$, $\alpha'(0) = \vec{v}$.

Then
$$F_* v = \frac{d}{dt}\Big|_{t=0} F(\alpha(t))$$

Pf $F = (F_1 — F_m)$

$$F_* \vec{v}_p = \left((F_1)_* \vec{v}_p, \ \ldots \ (F_m)_* \vec{v}_p \right)$$
$$= \left(\nabla F_1(p) \cdot \vec{v}, \ \ldots, \ \nabla F_m(p) \cdot \vec{v} \right)$$
$$= \left(\nabla F_1(\alpha(0)) \cdot \alpha'(0), \ \ldots, \ \nabla F_m(\alpha(0)) \cdot \alpha'(0) \right)$$
$$= \frac{d}{dt}\Big|_{t=0} \left(F_1(\alpha(t)), \ \ldots, \ F_m(\alpha(t)) \right)$$
$$= \frac{d}{dt}\Big|_{t=0} F(\alpha(t))$$