MATH 423 EXAM 2, SPRING 2010

SOLUTIONS

(1)

(a) $\omega_p(\vec{v}) = 4^2(-6) + 1\times 4(5) + 1\times 3(2) = -70.$

(b)
$$\eta_p(\vec{v},\vec{w}) = z(p)\left[dx(\vec{v})\,dy(\vec{w}) - dx(\vec{w})\,dy(\vec{v})\right] + x(p)\left[dy(\vec{v})\,dz(\vec{w}) - dy(\vec{w})\,dz(\vec{v})\right]$$
$$= 3\left[-6\times -2 - 0\times 5\right] + 1\left[5\times 7 - -2\times 2\right]$$
$$= 36 + 39 = 75$$

(c) SEE NEXT PAGE.

(d)
$$\omega = y^2\,dx + xy\,dy + xz\,dz$$
$$d\omega = d(y^2)\wedge dx + d(xy)\wedge dy + d(xz)\wedge dz$$
$$= 2y\,dy\wedge dx + y\,dx\wedge dy + z\,dx\wedge dz$$
$$= -2y\,dx\wedge dy + y\,dx\wedge dy \mp z\,dz\wedge dx$$
$$= -y\,dx\wedge dy + 0\,dy\wedge dz - z\,dz\wedge dx$$

1c

Suppose

$$a\,dx\wedge dy + b\,dy\wedge dz + c\,dz\wedge dx = 0 \quad \circledast$$

We must show $a = b = c = 0$.

Apply $\circledast$ to the pair of vectors U_1, U_2:

$$0 = a\,dx\wedge dy(U_1, U_2) + b\,dy\wedge dz(U_1, U_2) + c\,dz\wedge dx(U_1, U_2)$$

$$= a(1.1. - 0.0) + b(0.0 - 1.0) + c(0.0 - 0.0)$$

$$= a$$

So $a = 0$.

Similarly if apply $\circledast$ to (U_2, U_3) get

$$0 = 0 + b\left(dy(U_2)\,dz(U_3) - dy(U_3)\,dz(U_2)\right) + 0$$

$= b.1$ So $b = 0$

And finally if apply $\circledast$ to (U_3, U_1) you get

$c = 0$ too.

So $a = b = c = 0$ as required.

~~1d~~ 1e $\alpha^*\omega(\vec{v}) = \omega(\alpha_*(\vec{v}))$

$$(\alpha^*\omega)_t = \omega_{\alpha(t)}(\alpha'(t))\,dt$$

$$\alpha(t) = (\cos t, \sin t, t)$$

$$\alpha'(t) = (-\sin t, \cos t, 1)$$

$$\omega_{\alpha(t)}(\alpha'(t)) = \sin^2 t(-\sin t) + \cos t \sin t(\cos t) + t\cos t\,(1)$$

$$= -\sin^3 t + \cos^2 t \sin t + t \cos t$$

So

$$(\alpha^*\omega)_t = (-\sin^3 t + \cos^2 t \sin t + t\cos t)\,dt.$$

1f $\alpha^*\eta$ is a 2-form on $\mathbb{R}$.

Therefore it must be zero: $\alpha^*\eta = 0$.

REASON Let $\vec{v}, \vec{w} \in T_t\mathbb{R}$. Then for any 2-form Ω on $\mathbb{R}$ we have $\Omega(\vec{v}, \vec{w}) \in \mathbb{R}$. Now since $\dim T_t\mathbb{R} = 1$, $\vec{v}$ and $\vec{w}$ must be linearly independent.

We may as well assume that

$$\vec{w} = c\,\vec{v} \quad \text{for some } c \in \mathbb{R}.$$

(This will be true unless $\vec{v} = 0$ and $\vec{w} = 0$ in which case $\Omega(\vec{v},\vec{w}) = \Omega(0 \cdot \vec{v}, 0\,\vec{w}) = 0\,\Omega(\vec{v},0) = 0$)

So

$$\Omega(\vec{v},\vec{w}) = c\,\Omega(\vec{v},\vec{v}) \overset{(*)}{=} -c\,\Omega(\vec{v},\vec{v})$$

$$\cancel{= -\Omega(\vec{w},\vec{v}) = -\Omega(\vec{v},\vec{v})}$$

So if $c = 0$, $\Omega(\vec{v},\vec{w}) = 0$.

and if $c \neq 0$, $\Omega(\vec{v},\vec{v}) = -\Omega(\vec{v},\vec{v})$ so

$$\Omega(\vec{v},\vec{v}) = 0 \quad \text{so} \quad \Omega(\vec{v},\vec{w}) = 0.$$

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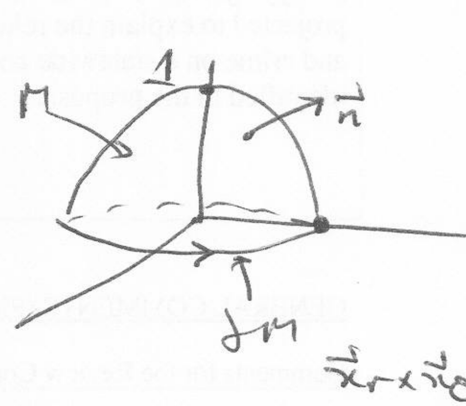
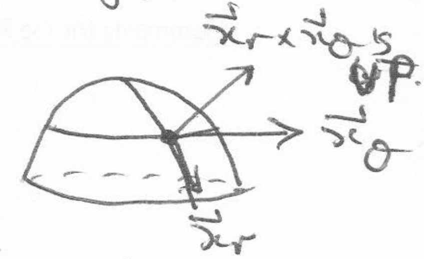
(2) Parametrize M using



$$\vec{x}(r,\theta) = (r\cos\theta,\ r\sin\theta,\ 1-r^2)$$

for $0 < r \leq 1$ and $0 \leq \theta < 2\pi$.



$$\vec{x}_r = \frac{\partial \vec{x}}{\partial r} = (\cos\theta,\ \sin\theta,\ -2r)$$

$\vec{x}_\theta = \frac{\partial \vec{x}}{\partial \theta}$ ... UP/OUTWARD NORMAL

Then

(a) $$\int_M d\omega = ?$$

Well $\omega = z\,dx + x\,dy$

So $d\omega = dz \wedge dx + dx \wedge dy$

So

$$\int_M d\omega = \int_{\theta=0}^{2\pi}\int_{r=0}^{1} \vec{x}^*(d\omega)$$

$$= \int_0^{2\pi}\int_0^{1} d\omega(\vec{x}_r, \vec{x}_\theta)\, dr\, d\theta$$

Now

$$d\omega(\vec{x}_r, \vec{x}_\theta) = dz(\vec{x}_r)\,dx(\vec{x}_\theta) - dz(\vec{x}_\theta)\,dx(\vec{x}_r)$$

$$+ dx(\vec{x}_r)\,dy(\vec{x}_\theta) - dx(\vec{x}_\theta)\,dy(\vec{x}_r)$$

$$= -2r(-r\sin\theta) - 0 . \cos\theta$$

$$+ \cos\theta\, r\cos\theta - (-r\sin\theta)\sin\theta$$

$$= 2r^2\sin\theta \ldots$$

So

$$\int_M d\omega = \int_0^{2\pi}\int_0^1 \left(2r^2\sin\theta + r\right) dr\, d\theta$$

$$= 2\pi \int_0^1 r\, dr$$

$$= 2\pi \left[r^2/2\right]_0^1 = \pi$$

(b) Parametrize ∂M by setting $\alpha(\theta) = \vec{x}(1,\theta)$

$$\alpha(\theta) = (\cos\theta, \sin\theta, 0), \quad 0 < \theta < 2\pi$$

This parametrization of ∂M curve has ~~correct~~ induced orientation from M. (See picture on P3)

Then by FTC

$$\alpha(\theta) = (\cos\theta, \sin\theta, 0)$$

$$\alpha'(\theta) = (-\sin\theta, \cos\theta, 0)$$

$$\omega = z\,dx + x\,dy$$

$$\int_M d\omega = \int_{\partial M} \omega$$

$$= \int_0^{2\pi} \alpha^* \omega$$

$$= \int_0^{2\pi} \omega_{\alpha(\theta)}(\alpha'(\theta))\, d\theta$$

$$= \int_0^{2\pi} 0.(-\sin\theta) + \cos\theta \cos\theta\, d\theta$$

$$= \int_0^{2\pi} \cos^2\theta \, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} (1 + \cos 2\theta)\, d\theta$$

$$= \frac{1}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{2\pi} = \pi$$

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3 $\vec{x}(u,v) = (u, v, f(u,v))$

$$\vec{x}_u = (1, 0, f_u)$$

$$\vec{x}_v = (0, 1, f_v)$$

$\vec{x}_u \times \vec{x}_v = (-f_u, -f_v, 1)$ is upward.

$$E = \vec{x}_u \cdot \vec{x}_u = 1 + f_u^2$$

$$F = \vec{x}_u \cdot \vec{x}_v = f_u f_v$$

$$G = \vec{x}_v \cdot \vec{x}_v = 1 + f_v^2$$

$$W = \sqrt{EG - F^2} = \sqrt{(1+f_u^2)(1+f_v^2) - f_u^2 f_v^2}$$

$$= \sqrt{1 + f_u^2 + f_v^2}$$

$$U = \frac{\vec{x}_u \times \vec{x}_v}{W} = \frac{1}{W}(-f_u, -f_v, 1)$$

$$L = U \cdot \vec{x}_{uu} \qquad \vec{x}_{uu} = (0, 0, f_{uu})$$ ⑦

$$M = U \cdot \vec{x}_{uv} \qquad \vec{x}_{uv} = (0, 0, f_{uv})$$

$$N = U \cdot \vec{x}_{vv} \qquad \vec{x}_{vv} = (0, 0, f_{vv})$$

So

$$L = \frac{1}{W}(-f_u, -f_v, 1) \cdot (0, 0, f_{uu}) = \frac{f_{uu}}{W}$$

$$M = \frac{1}{W}(-f_u, -f_v, 1) \cdot (0, 0, f_{uv}) = \frac{f_{uv}}{W}$$

$$N = \frac{1}{W}(-f_u, -f_v, 1) \cdot (0, 0, f_{vv}) = \frac{f_{vv}}{W}.$$

$$K = \frac{LN - M^2}{EG - F^2} = \frac{\dfrac{f_{uu} f_{vv} - f_{uv}}{W^2}}{W^2}$$

$$= \frac{f_{uu} f_{vv} - f_{uv}}{W^2} = \frac{f_{uu} f_{vv} - f_{uv}}{(1 + f_u^2 + f_v^2)^2}$$

$$H = \frac{GL + EN - 2FM}{2(EG - F^2)}$$

$$= \frac{(1 + f_v^2) f_{uu} + (1 + f_u^2) f_{vv} - 2 f_u f_v f_{uv}}{2W^3}$$

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$$H = \frac{(1+f_v^2) f_{uu} + (1+f_u^2) f_{vv} - 2 f_u f_v f_{uv}}{2\,(1+f_u^2+f_v^2)^{3/2}}$$

3b

$f(u,v) = uv \qquad p = (2,3,6)$

$f_u = v \qquad f_u(p) = 3$

$f_v = u \qquad f_v(p) = 2$

$f_{uu} = 0 \qquad f_{uu}(p) = 0$

$f_{vv} = 0 \qquad f_{vv}(p) = 0$

$f_{uv} = 1 \qquad f_{uv}(p) = 1$

$$K = \frac{0 \times 0 - 1}{(1+3^2+2^2)^2} = \frac{-1}{\cancel{196}\, 14^2} = \frac{-1}{196}$$

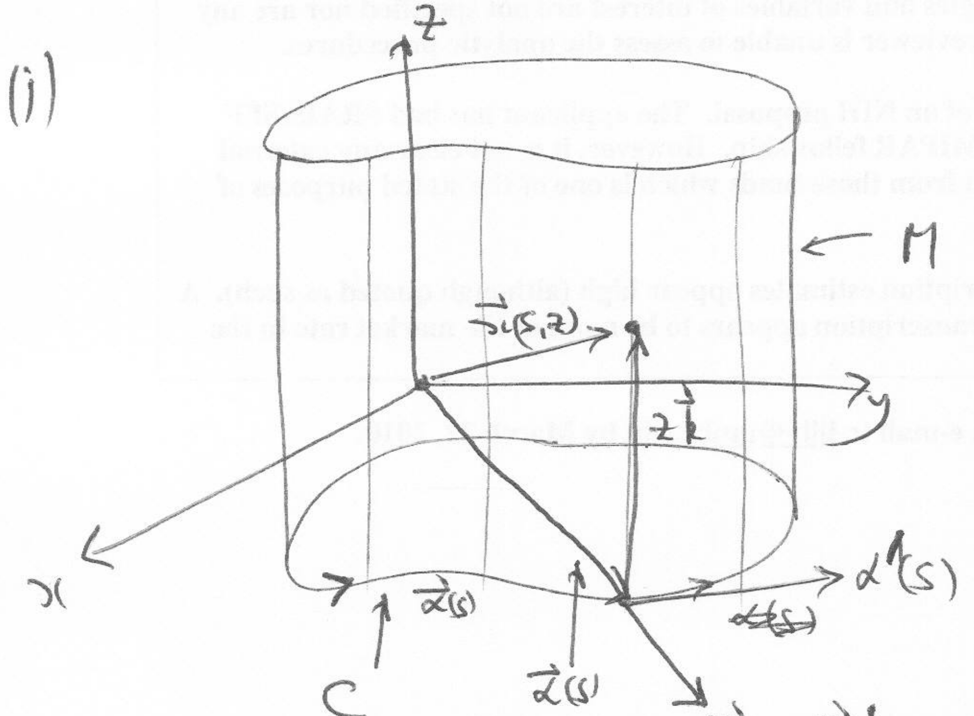
$$H = \frac{(1+2^2)0 + (1+3^2)0 - 2 \times 3 \times 2 \times 1}{2\,(1+3^2+2^2)^{3/2}} = \frac{-12}{2(14)^{3/2}}$$

$$= \frac{-6}{}$$

(4) (a) $\vec{x}(s,z) = \big(x(s),\ y(s),\ z\big)$

$= \vec{\alpha}(s) + z\vec{k}$

(i)



$\vec{n} = \vec{\alpha}'(s) \times \vec{k}$ is OUTWARD.

This picture shows $\vec{x}(s,z)$ is a point of M.

So $\vec{x} : \overbrace{(a,b) \times \mathbb{R}}^{\text{OPEN IN } \mathbb{R}^2} \longrightarrow M.$

$\vec{x}$ omits the vertical line from M that goes through $\vec{\alpha}(a)$. It covers the rest of M.

(ii) $\underline{\vec{x} \text{ is } 1\text{-}1}$

Suppose $\vec{x}(s,z) = \vec{x}(t,v)$

So

$$(x(s), y(s), z) = (x(t), y(t), w)$$

So by 3rd component $z = w$

and by 1st two components

$$\alpha(s) = \alpha(t) \qquad a < s \le t < b$$

So by def^n of α as a simple curve

we must have $s = t$

So $(s, z) = (t, w)$

So $\vec{x}$ is 1–1

(iii) $\underline{\vec{x} \text{ is regular}}$

$\vec{x}_s = \vec{\alpha}'(s)$

$\vec{x}_z = \vec{k}$

Since $\vec{\alpha}$ is a unit speed curve in $\mathbb{R}^2$

$\vec{\alpha}'(s)$ is a non zero vector in xy-plane.

4b From the picture on 9 we see that

$$\vec{x}_s \times \vec{x}_z = \alpha'(s) \times \vec{k} \text{ is outward normal.}$$

Now

$$\|\vec{x}_s \times \vec{x}_z\| = \|\alpha'(s)\| \, \|\vec{k}\| \sin \pi/2 = 1 \cdot 1 \cdot 1 = 1$$

as α is unit speed in the xy-plane

So

$$\mathbb{U} = \vec{x}_s \times \vec{x}_z = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x'(s) & y'(s) & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= (y'(s), \; -x'(s), \; 0)$$

4c

$$S(\vec{x}_s) = -\nabla_{\vec{x}_s} \mathbb{U}$$

$$= -\frac{d}{ds}\left(\mathbb{U}(\alpha(s))\right) = -\frac{d}{ds} \mathbb{U}(s)$$

$$= -\frac{d}{ds}\left(+y'(s), \; -x'(s), \; 0\right)$$

$$= \left(-y''(s), \; x''(s), \; 0\right)$$

And

$$S(\vec{x}_z) = S(\vec{k}) = -\nabla_{\vec{k}} U = -\frac{\partial}{\partial z} U = \vec{0}.$$

Now $\{\vec{x}_s, \vec{x}_z\}$ is an <u>orthonormal basis</u> for T_pM (by 4b) and so

$$S(\vec{x}_s) = S(\vec{x}_s)\cdot\vec{x}_s\,\vec{x}_s + S(\vec{x}_s)\cdot\vec{x}_z\,\vec{x}_z$$

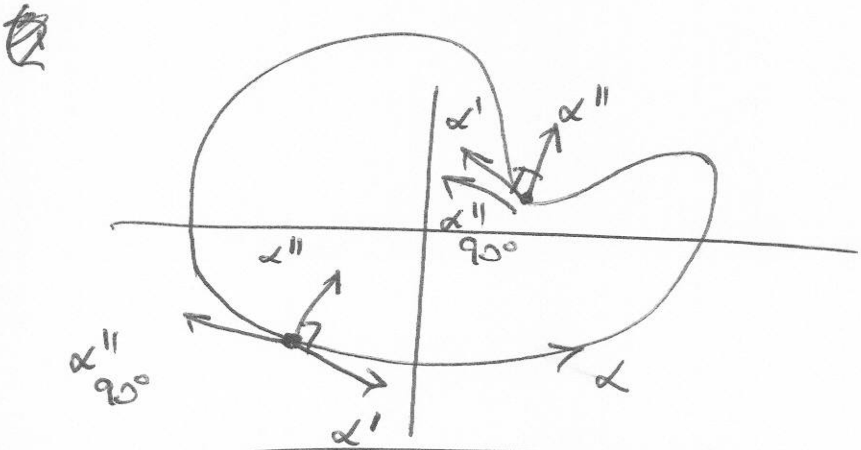
$$= (-y'', x'', 0)\cdot(x', y', 0)\,\vec{x}_s + (-y'', x'', 0)\cdot(0, 0, 1)\,\vec{x}_z$$

$$= (-x'y'' + y'x'')\,\vec{x}_s + 0\,\vec{x}_z$$

So matrix of S is

$$[S] = \begin{bmatrix} -x'y'' + y'x'' & 0 \\ 0 & 0 \end{bmatrix}$$

$$\boxed{\text{What is geometrical meaning of } -x'y'' + y'x''\text{?}}$$



Let $\alpha''_{90^\circ} = (-y'', x'')$.

Notice that

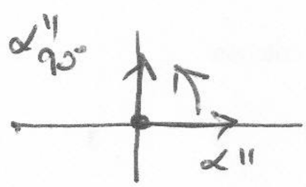
$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{pmatrix} -y'' \\ x'' \end{pmatrix},$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\alpha'' = \alpha''_{90^\circ}$$

$\alpha' \cdot \alpha'' = 0$ as see from

So $\alpha''_{90^\circ}(t)$ is a 90° rotation of $\alpha''(t)$ in $\mathbb{R}^2$. (about point $\alpha(t)$).

To check which direction this rotation is notice that if $\begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ Then $\begin{pmatrix} -y'' \\ x'' \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$



So it is a CLOCKWISE rotation by 90°.

So from the picture we see that α''_{90° and α' are either parallel or antiparallel depending on concavity of curve (ie on whether α'' points in/out of region bounded by C)

So

$$-y''x' + y'x'' = \alpha''_{90^\circ} \cdot \alpha' = \|\alpha''_{90^\circ}\| \|\alpha'\| \cos\theta$$

where $\theta = 0$ or π (See above)

So as $\|\alpha''_{90^\circ}\| = \|\alpha''\|$ and $\|\alpha'\| = 1$ we get

$$-y''x' + y'x'' = \begin{cases} -\|\alpha''\| & \text{if } \alpha'' \text{ points in} \\ \|\alpha''\| & \text{if } \alpha'' \text{ points out} \end{cases}$$

So

$$[S] = \begin{bmatrix} \pm\|\alpha''\| & 0 \\ 0 & 0 \end{bmatrix}$$ with sign given on bottom of 15

4d $k_1 = \pm\|\alpha''\| \qquad \vec{V}_1 = \alpha'$

$k_2 = 0 \qquad \vec{V}_2 = \vec{k}$

$K = k_1 k_2 = 0$

$H = \frac{1}{2}(k_1 + k_2) = \dfrac{\pm\|\alpha''\|}{2}.$

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(4e) When M is $x^2+y^2=R^2$ we have

$$\alpha(s) = R\left(\cos(s/R), \sin(s/R)\right) \qquad 0<s<2\pi$$

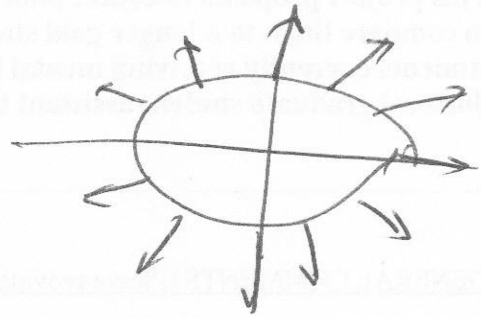
So $\alpha'(s) = \left(-\sin(s/R), \cos(s/R)\right)$

has $\|\alpha'(s)\| = 1$ as required.

(i) And from (b)

$$U = \left(\cos(s/R), \sin(s/R), 0\right)$$

This is a length 1 vector field. It lies in horizontal plane and points radially out from z axis



So it is indeed the unit outward normal to M.

(ii) $$S = \begin{bmatrix} \dfrac{-\sin^2(s/R)}{R} - \dfrac{\cos^2(s/R)}{R} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{R} & 0 \\ 0 & 0 \end{bmatrix}$$

as $x'(s) = -\sin(s/R)$ $\qquad y'(s) = \cos(s/R)$

$x''(s) = -\frac{1}{R}\cos(s/R)$ $\qquad y''(s) = -\frac{1}{R}\sin(s/R)$

So $k_1 = -\frac{1}{R} < 0$

$\vec{V}_1 = \vec{x}_s = \alpha'(s)$ is tangent to circle

$k_2 = 0$ $\qquad K = 0$

$\vec{V}_2 = \vec{k}$ $\qquad H = -\frac{1}{2R}$

This agrees with previous calculations we did in class.

<u>NOTE</u> The fact $k_1 < 0$ makes sense as the circle on cylinder bends away from the outward normal.