NAME:

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MATH 423/673 (Spring 2010) Exam 2, May 1st-3rd

Instructions

This take home exam is due 12 noon Monday May 3rd in the Math Office Mail Slot on the 4th floor of Math/Psych Building. Put your solutions in a yellow envelope addressed to me (John Zweck) and make sure all other names and addresses on the envelope are crossed out. No late submissions will be accepted without prior approval from me (ie before 12 noon Monday May 3rd), and such exceptions will only be made in documented emergency situations.

This exam is open book, open notes, open other written sources, but you are not allowed to talk or communicate in any way with anyone about the exam or Math 423/673 in general, either in person or via electronic communication of any sort, until after 12 noon Monday May 3rd. The one exception to this rule is that you may email me questions.

Show all work and give **complete explanations** for all your answers.

Write your solutions on separate sheets of paper, number the pages and the problems, staple all pages with the exam question sheets on the top. Write your name in the box at start of this page and sign the pledge:

Pledge: *I have neither given nor received aid on this exam*

Signature: ______________________________

(1) [20 pts] Let ω be the 1-form on $\mathbf{R}^3$ given by

$$\omega = y^2\,dx + xy\,dy + xz\,dz,$$

and let η be the 2-form on $\mathbf{R}^3$given by

$$\eta = z\,dx \wedge dy + x\,dy \wedge dz.$$

(a) Calculate $\omega_p(\mathbf{v})$, where $p = (1, 4, 3)$ and $\mathbf{v} = (-6, 5, 2)$.

(b) Calculate $\eta_p(\mathbf{v}, \mathbf{w})$, where $p = (1, 4, 3)$, $\mathbf{v} = (-6, 5, 2)$, and $\mathbf{w} = (0, -2, 7)$.

(c) Prove that $\{dx \wedge dy, dy \wedge dz, dz \wedge dx\}$ is a linearly independent set of 2-forms on $\mathbf{R}^3$.

(d) Express the 2-form $d\omega$ in the standard basis $dx \wedge dy$, $dy \wedge dz$, $dz \wedge dx$ for the vector space of 2-forms on $\mathbf{R}^3$.

(e) Let $\alpha : \mathbf{R} \to \mathbf{R}^3$ be the curve given by

$$\alpha(t) = (\cos t, \sin t, t).$$

Calculate $\alpha^*(\omega)$.

(f) Without explicitly calculating it, what can you say about $\alpha^*(\eta)$? Why?

(2) [15 pts] Let M be the oriented surface with boundary that is given by

$$\begin{aligned} z &= 1 - x^2 - y^2, \\ z &> 0, \end{aligned}$$

with the upward orientation. (So the boundary, ∂M, is the curve $x^2 + y^2 = 1$ in the plane $z = 0$.) Let ω be the 1-form on $\mathbf{R}^3$ given by

$$\omega = z\,dx + x\,dy.$$

(a) Calculate $\int_M d\omega$ directly (by integrating the 2-form, $d\omega$, over the surface, M).

(b) Check your answer by converting $\int_M d\omega$ to an integral over ∂M.

(3) [15 pts]
(a) Let M be an oriented surface that is the image of a Monge patch $\mathbf{x}(u,v) = (u, v, f(u,v))$, with the upward orientation. Show that

$$\begin{aligned} E &= 1 + f_u^2, & L &= \frac{f_{uu}}{W}, \\ F &= f_u f_v, & M &= \frac{f_{uv}}{W}, \\ G &= 1 + f_v^2, & N &= \frac{f_{vv}}{W}, \end{aligned}$$

where

$$W = \sqrt{EG - F^2} = (1 + f_u^2 + f_v^2)^{1/2}.$$

Then find formulae for the Gauss curvature, K, and the mean curvature, H.

(b) Using (a), calculate K and H at the point $(2, 3, 6)$ on the saddle surface $z = xy$.

(4) [25 pts] Let $\alpha : [a, b] \to \mathbf{R}^2$ be a unit speed parametrization of a simple closed curve, C, in the plane. [Recall that α is closed if $\alpha(a) = \alpha(b)$ and that α is simple if it is non-self intersecting, in that if $\alpha(x) = \alpha(y)$ and $a < x \leq y < b$ then $x = y$. A circle is a simple closed curve. A figure-eight curve is closed but is not simple.]
Let M be the surface obtained by translating the curve C perpendicular to the plane. (We call M a *generalized cylinder*.)

(a) Suppose $\alpha(s) = (x(s), y(s))$, for some functions $x(s)$ and $y(s)$. Show that

$$\mathbf{x}(s, z) = (x(s), y(s), z), \qquad s \in (a, b), z \in \mathbf{R},$$

is a patch for M.

(b) Suppose that the curve C in the xy-plane is oriented in the counter-clockwise direction when viewed from above (ie from the positive z-axis). Show that the outward pointing unit normal to M is

$$U = \left(\frac{\partial y}{\partial s}, -\frac{\partial x}{\partial s}, 0 \right).$$

(c) Calculate the matrix of the shape operator, S, in the basis $\{\mathbf{x}_s, \mathbf{x}_z\}$ for T_pM.

(d) Hence calculate the principal curvatures and directions as well as the Gauss and mean curvature of M.

(e) Verify that the formulae you derived in (b), (c), and (d) are correct in the special case that M is the circular cylinder of radius R, *i.e.*, $x^2 + y^2 = R^2$.