NAME: SOLUTIONS

1 /6	2 /17	3 /22	4 /10	5 /20	T /75

MATH 423 (Spring 2012) Exam 1, March 6th

No calculators, books or notes!
Show all work and give **complete explanations** for all your answers.

(1) [6 pts] Explain why the following equations are nonsensical. [Hint: Type Check!]

(a) $(f \circ \alpha)'(0) = \nabla f$, where α parametrizes a curve in $\mathbb{R}^3$ and $f : \mathbb{R}^3 \to \mathbb{R}$.

$f \circ \alpha : \mathbb{R} \to \mathbb{R}$ So $(f \circ \alpha)'(0)$ is a real #

but ∇f is a vector field on $\mathbb{R}^3$

(b) $F_*(\mathbf{v}_p) = 3U_1(p) + 4U_2(p) + 5U_3(p)$, where $F : \mathbb{R}^3 \to \mathbb{R}^2$.

$F_*(\vec{v}_p) \in T_{F(p)} \mathbb{R}^2$

but $3U_1(p) + 4U_2(p) + 5U_3(p) \in T_p \mathbb{R}^3$

(c) $\mathbf{v}_p[fg] = \mathbf{v}_p[f]g + \mathbf{v}_p[g]f$, where $\mathbf{v}_p$ is a tangent vector to $\mathbb{R}^3$ at a point p and $f, g : \mathbb{R}^3 \to \mathbb{R}$.

$\forall h$: $v_p[h] = \frac{d}{dt}\Big|_{t=0} (h(\vec{p} + t\vec{v}))$ is a real #.

So LHS is a real #

But f, g are functions. So RHS is a function on $\mathbb{R}^3$

CORRECT FORMULA $\vec{v}_p[fg] = v_p[f]\, g(p) + v_p[g]\, f(p)$

(2) [17 pts]

(a) State the definition of the directional derivative, $\mathbf{v}_p[f]$, of a function $f : \mathbb{R}^3 \to \mathbb{R}$ with respect to a tangent vector $\mathbf{v}_p$ to $\mathbb{R}^3$ at a point p.

Let $\alpha(t) = \vec{p} + t\vec{v}$ be line thru p in direction $\vec{v}$

Then $\vec{v}_p[f] = (f \circ \alpha)'(0)$

(b) Use the definition you gave in (a) to calculate $\mathbf{v}_p[f]$ for $f(x, y, z) = x^2 + z$ with respect to the vector $\mathbf{v} = (1, 0, 2)$ at the point $p = (1, 2, 3)$.

$$\alpha(t) = (1,2,3) + t(1,0,2)$$
$$= (1+t, 2, 3+2t)$$

$$(f \circ \alpha)(t) = (1+t)^2 + 3 + 2t$$
$$(f \circ \alpha)'(t) = 2(1+t) + 2 = 4 + 2t$$
$$\vec{v}_p[f] = (f \circ \alpha)'(0) = 4$$

(c) Starting from the definition you gave in (a) prove that $\mathbf{v}_p[f] = \nabla f(p) \cdot \mathbf{v}$.

$$\vec{v}_p[f] = (f \circ \alpha)'(0)$$

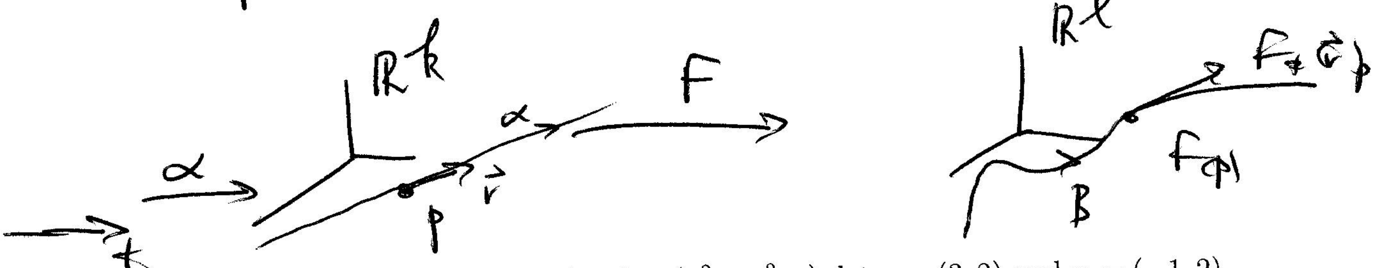
$= \nabla f(\alpha(0)) \cdot \alpha'(0)$ by Chain Rule for functions on curves

$= \nabla f(\vec{p}) \cdot \vec{v}$ as $\alpha(t) = \vec{p} + t\vec{v}$ so $\alpha(0) = \vec{p}$, $\alpha'(0) = \vec{v}$

(3) [22 pts]
(a) Define the tangent mapping of a mapping $F : \mathbb{R}^k \to \mathbb{R}^\ell$.

$F_* : T_p\mathbb{R}^k \longrightarrow T_{F(p)}\mathbb{R}^l$ is defined by

$F_*(\vec{v}_p) = \beta'(0)$ where $\beta = F \circ \alpha$, $\alpha(t) = \vec{p} + t\vec{v}$.



(b) Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $F(x, y) = (x^2 + y^2, x)$, let $p = (3, 2)$ and $\mathbf{v} = (-1, 2)$.

(i) Calculate $F_*(\mathbf{v}_p)$ using the definition you gave in (a).

$$\alpha(t) = (3 - t, 2 + 2t)$$

$$\beta(t) = F(\alpha(t)) = \left((3-t)^2 + (2+2t)^2, 3 - t\right)$$

$$\beta'(t) = \left(-2(3-t) + 2(2+2t)2, -1\right)$$

$$\beta'(0) = (-6 + 4\times 2, -1) = (2, -1)$$

So $F_*(\vec{v}_p) = \beta'(0) = (2, -1)$

(ii) Calculate $F_*(\mathbf{v}_p)$ using the matrix of partial derivatives of F.

$$DF(x,y) = \begin{pmatrix} 2x & 2y \\ 1 & 0 \end{pmatrix}$$

$$DF(p) = \begin{pmatrix} 6 & 4 \\ 1 & 0 \end{pmatrix}$$

$$F_*(\vec{v}_p) = [DF(p)]\,\vec{v} = \begin{pmatrix} 6 & 4 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

(iii) Use the Inverse Function Theorem to show that F is locally invertible at p and calculate $(F^{-1})_*\mathbf{w}_{F(p)}$, where $\mathbf{w} = (4, -3)$ and $p = (3, 2)$.

Must show $(F_*)_p : T_p\mathbb{R}^2 \to T_{F(p)}\mathbb{R}^2$ is invertible. Since matrix of $(F_*)_p$ is

$DF(p) = \begin{pmatrix} 6 & 4 \\ 1 & 0 \end{pmatrix}$ is invertible

$(\det DF(p) = -4 \neq 0)$ we conclude $(F_*)_p$ is invertible and so F is locally invertible at p

$$(F^{-1})_*(\vec{w}_{F(p)}) = [D(F^{-1})(F(p))]\,\vec{w} \overset{IFT}{=} [DF(p)]^{-1}\vec{w}$$

$$= \begin{pmatrix} 6 & 4 \\ 1 & 0 \end{pmatrix}^{-1}\begin{pmatrix} 4 \\ -3 \end{pmatrix} = -\frac{1}{4}\begin{pmatrix} 0 & -4 \\ -1 & 6 \end{pmatrix}\begin{pmatrix} 4 \\ -3 \end{pmatrix} = \begin{pmatrix} -3 \\ -11/2 \end{pmatrix}$$

(4) [10 pts] Let $\mathbf{v}$ and $\mathbf{w}$ be vectors in $\mathbb{R}^3$ with $\mathbf{v} \neq 0$ and let $\alpha : \mathbb{R} \to \mathbb{R}^3$ be the curve defined by $\alpha(t) = t^3\mathbf{v} + \mathbf{w}$.

(a) Let $t_* \in \mathbb{R}$. Calculate the length of α from $t = 0$ to $t = t_*$.

$$L(t_*) = \int_0^{t_*} \|\alpha'(u)\| \, du$$

$$= \int_0^{t_*} \|3u^2 \vec{v}\| \, du$$

$$= \int_0^{t_*} 3u^2 . \|\vec{v}\| \, du = \|\vec{v}\| \int_0^{t_*} 3u^2 \, du$$

$$= t_*^3 \|\vec{v}\|$$

(b) Hence calculate a unit-speed reparameterization of α.

$$s = t^3 \|\vec{v}\| = s(t)$$

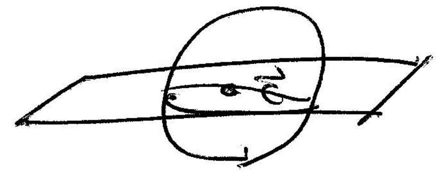
$$t = \left(\frac{s}{\|\vec{v}\|}\right)^{1/3} = t(s)$$

$$\beta(s) = \alpha(t(s)) = \alpha\left(\left(\frac{s}{\|\vec{v}\|}\right)^{1/3}\right)$$

$$= \frac{s}{\|\vec{v}\|} \vec{v} + \vec{w}.$$

Unit, $\beta'(s) = \frac{\vec{v}}{\|\vec{v}\|}$ has length 1

$\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ IS ANY ONB
not necessarily $\vec{i}, \vec{j}, \vec{k}$



(5) [20 pts] Let $r > 0$, let $\mathbf{c}$ be a point in $\mathbb{R}^3$ and let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be an orthonormal basis for the tangent space to $\mathbb{R}^3$ at $\mathbf{c}$. Let γ be the curve defined by

$$\gamma(s) = \mathbf{c} + r\cos(s/r)\mathbf{e}_1 + r\sin(s/r)\mathbf{e}_2.$$

(a) Show that γ parametrizes the circle of radius r centered at $\mathbf{c}$ that lies in the plane containing the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$. [Hints: Use $\|a_1\mathbf{e}_1 + a_2\mathbf{e}_2\| = \sqrt{a_1^2 + a_2^2}$ to calculate the distance from an arbitrary point on the curve γ to $\mathbf{c}$. Also explain why $\gamma(s)$ lies in the plane through $\mathbf{c}$ that contains the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$.]

$$\|\gamma(s) - \vec{c}\| = \left\| r\cos\left(\frac{s}{r}\right)\vec{e}_1 + r\sin\left(\frac{s}{r}\right)\vec{e}_2 \right\|$$

$$= \sqrt{r^2\cos^2\left(\frac{s}{r}\right) + r^2\sin^2\left(\frac{s}{r}\right)} \quad \text{by Hint}$$

$$= r$$

So $\gamma(s)$ lies on SPHERE center $\vec{c}$ radius r

Also $\gamma(s) = \vec{c} + a\vec{e}_1 + b\vec{e}_2$ where $a = r\cos\left(\frac{s}{r}\right)$, $b = r\sin\left(\frac{s}{r}\right)$

So $\gamma(s)$ is in plane thru $\vec{c}$ containing $\vec{e}_1, \vec{e}_2$.

The intersection of this plane + sphere is the circle radius r center $\vec{c}$

(b) Show that γ is a unit speed curve.

$$\gamma'(s) = -\sin\left(\frac{s}{r}\right)\vec{e}_1 + \cos\left(\frac{s}{r}\right)\vec{e}_2$$

So by hint above

$$\|\gamma'(s)\| = \sqrt{\sin^2(s/r) + \cos^2(s/r)} = 1$$

(c) Calculate the Frenet frame, T, N, and B, of the curve γ. Hence find the curvature and torsion of γ. [Hint: Express T, N, and B in terms of s, r, $\mathbf{e}_1$ and $\mathbf{e}_2$.]

$$T = \gamma' = -\sin\left(\frac{s}{r}\right)\vec{e}_1 + \cos\left(\frac{s}{r}\right)\vec{e}_2$$

$$T' = \gamma'' = -\frac{1}{r}\cos\left(\frac{s}{r}\right)\vec{e}_1 - \frac{1}{r}\sin\left(\frac{s}{r}\right)\vec{e}_2$$

$$K = \|T'\| = \frac{1}{r} \quad \text{by HINT to (a)}$$

$$N = \frac{T'}{K} = -\cos(s/r)\vec{e}_1 - \sin\left(\frac{s}{r}\right)\vec{e}_2$$

$$B = T \times N$$

$$= \left(-\sin\left(\frac{s}{r}\right)\vec{e}_1 + \cos\left(\frac{s}{r}\right)\vec{e}_2\right) \times \left(-\cos\left(\frac{s}{r}\right)\vec{e}_1 - \sin\left(\frac{s}{r}\right)\vec{e}_2\right)$$

$$= \sin^2\left(\frac{s}{r}\right)\vec{e}_1 \times \vec{e}_2 - \cos^2(s/r)\,\vec{e}_2 \times \vec{e}_1$$

$$= \vec{e}_1 \times \vec{e}_2. \quad \text{(CONSTANT)}$$

$$B' = 0 \quad \text{So} \quad B' = -\tau N, \; N \neq \vec{0} \Rightarrow \tau = 0$$

Pledge: *I have neither given nor received aid on this exam*

Signature: __________________________