NAME: SOLUTIONS

1 /12	2 /16	3 /12	4 /15	5 /8	6 /12	T /75

MATH 423 (Spring 2012) Exam II, April 12th

No calculators, books or notes!
Show all work and give **complete explanations** for all your answers.

(1) [12 pts]
(a) Define the concept of a 1-form on $\mathbf{R}^3$.

A 1-form ω on $\mathbb{R}^3$ is a choice of linear transformation $\omega_p : T_p\mathbb{R}^3 \to \mathbb{R}$ for each $p \in \mathbb{R}^3$.

ie If $\vec{v} \in T_p\mathbb{R}^3$ then $\omega_p(\vec{v}) \in \mathbb{R}$ and

$$\omega_p(c_1\vec{v}_1 + c_2\vec{v}_2) = c_1\omega_p(\vec{v}_1) + c_2\omega_p(\vec{v}_2) \quad \forall c_j \in \mathbb{R}, \vec{v}_j \in T_p\mathbb{R}^3$$

(b) Let V be a vector field on on $\mathbf{R}^3$. Define ω by $\omega_p(\mathbf{w}) = \mathbf{w} \cdot V(p)$, where p is a point in $\mathbf{R}^3$ and $\mathbf{w}$ is a tangent vector to $\mathbf{R}^3$ at p. Prove that ω is a 1-form on $\mathbf{R}^3$.

$\omega_p(\vec{w}) \in \mathbb{R}$, So $\omega_p : T_p\mathbb{R}^3 \to \mathbb{R}$.

Check linearity

$$\omega_p(c_1\vec{w}_1 + c_2\vec{w}_2) = (c_1\vec{w}_1 + c_2\vec{w}_2) \cdot V(p)$$

$$= c_1\vec{w}_1 \cdot V(p) + c_2\vec{w}_2 \cdot V(p)$$

$$= c_1\omega_p(\vec{w}_1) + c_2\omega_p(\vec{w}_2) \quad ✓$$

(2) [16 pts]

(a) Define the concept of a *coordinate patch.*

A coordinate patch is a 1–1, regular, mapping $\vec{x} : D \to \mathbb{R}^3$ where D is an open subset of $\mathbb{R}^2$.

$\vec{x}$ regular means $\vec{x}_* : T_p\mathbb{R}^2 \to T_{\vec{x}(p)}\mathbb{R}^3$ is 1–1 $\forall p \in D$.

(b) Does the equation $(x^2+y^2)^2 + 4z^2 - 16z + 15 = 0$ define a surface?

$F(x,y,z) = (x^2+y^2)^2 + 4z^2 - 16z + 15$

$\nabla F = \left(4x(x^2+y^2),\ 4y(x^2+y^2),\ 8z-16\right) = (0,0,0)$

at $(x,y,z) = (0,0,2)$.

Since $F(0,0,2) = 4.4 - 16.2 + 15 = -1 \neq 0$

we conclude $\nabla F \neq \vec{0}$ on $F=0$

So by Implicit Function Thm $\{F=0\}$ is a surface.

<u>Also</u> Need to ensure $\{F=0\}$ is non empty. Regard F as quadratic in z. Do there $\exists$ sol^{ns}?

$\Delta = b^2 - 4ac = (-16)^2 - 4.4.(15+(x^2+y^2)^2) = 16(16-15-(x^2+y^2)^2) \geq$

(3) [12 pts] Let M be the surface parametrized by $\mathbf{x}(u,v) = (u\cos v, u\sin v, v)$ for $0 < u < 1$ and $0 < v < \pi$. Orient M using the *upward* normal. Let $\mathbf{F}$ be the vector field on $\mathbf{R}^3$ defined by $\mathbf{F}(x,y,z) = (y,-x,z^2)$. Calculate $\iint_M \mathbf{F}\cdot d\mathbf{A}$.

$$\iint_M \vec{F}\cdot d\vec{A} = \int_{u=0}^{1}\int_{v=0}^{\pi} \vec{F}(u\cos v, u\sin v, v)\cdot\left(\frac{\partial\vec{x}}{\partial u}\times\frac{\partial\vec{x}}{\partial v}\right)dv\,du$$

Now

$$\frac{\partial\vec{x}}{\partial u}\times\frac{\partial\vec{x}}{\partial v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & 1 \end{vmatrix}$$

$$= (\sin v, -\cos v, u)$$

Since 3rd cpt is $u > 0$ we know $\frac{\partial\vec{x}}{\partial u}\times\frac{\partial\vec{x}}{\partial v}$ points upwards ✓.

So

$$\iint_M \vec{F}\cdot d\vec{A} = \int_{u=0}^{1}\int_{v=0}^{\pi}(u\sin v, -u\cos v, v^2)\cdot(\sin v, -\cos v, u)\,dv\,du$$

$$= \int_{u=0}^{1}\int_{v=0}^{\pi} u(\sin^2 v + \cos^2 v) + v^2 u\,dv\,du$$

$$= \int_{u=0}^{1}\int_{v=0}^{\pi} +u\,\cancel{\cos 2}v + v^2 u\,dv\,du = \int_0^1 u\,du\int_0^{\pi}(1+v^2)\,dv$$

$$= \frac{1}{2}\left[v + v^3/3\right]_0^{\pi} = \frac{\pi + \pi^3/3}{}$$

(4) [15 pts] The *catenoid*, M, is the surface obtained by rotating the catenary curve $z = \cosh y$ about the y-axis. [Recall that $\cosh x = (e^x + e^{-x})/2$ and $\sinh x = (e^x - e^{-x})/2$.]

(a) Find a formula for a coordinate patch $\mathbf{x} : D \to \mathbf{R}^3$ for M. Make sure you specify the domain D of $\mathbf{x}$. [Do *not* prove that $\mathbf{x}$ is a patch, just write down a formula for it.]

$$x = \cosh u \cos v$$

$$y = u$$

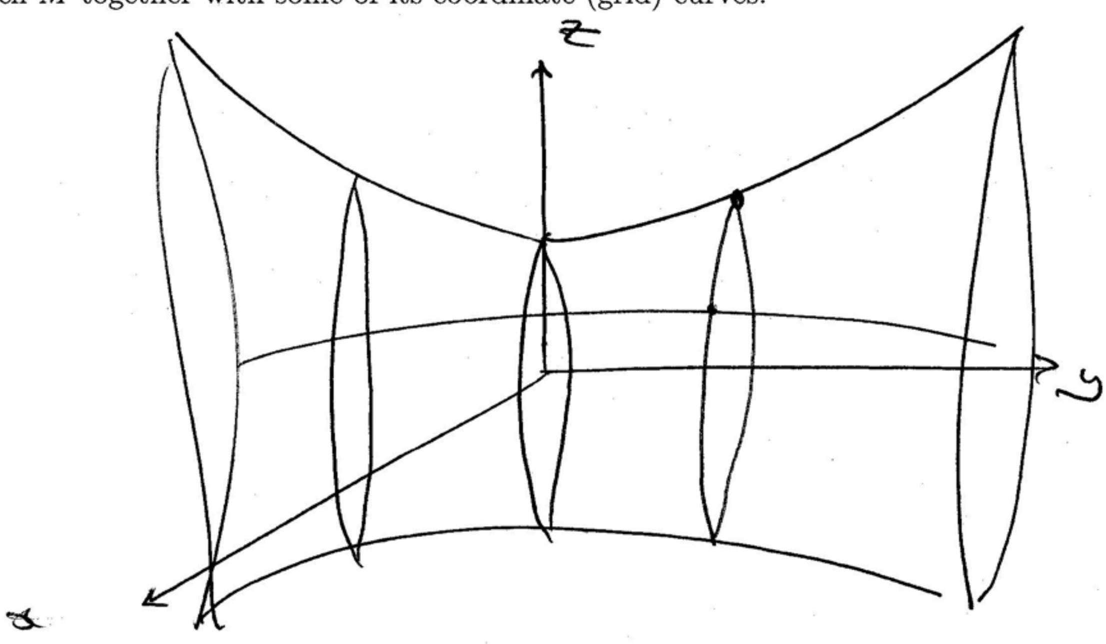
$$z = \cosh u \sin v$$

$u \in \mathbb{R}$

$v \in (0, 2\pi)$

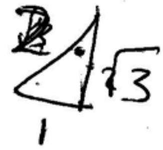
$$\alpha(u) = (u, \cosh u)$$

(b) Sketch M together with some of its coordinate (grid) curves.



(c) Calculate a parametrization of the tangent plane to M at the point $(x, y, z) = (\frac{\sqrt{3}}{2}\cosh 1, 1, \frac{1}{2}\cosh 1)$.

$u = 1, \quad v = \pi/6$



$$\vec{x}_u = (\sinh u \cos v,\ 1,\ \sinh u \sin v)$$

$$= \left(\sinh 1 \tfrac{\sqrt{3}}{2},\ 1,\ (\sinh 1)\tfrac{1}{2}\right)$$

$$\vec{x}_v = (-\cosh u \sin v,\ 0,\ \cos v \cosh u) = \left(\tfrac{-1}{2}\cosh 1,\ 0,\ \tfrac{\sqrt{3}}{2}\cosh\right.$$

$$L(s,t) = \left(\tfrac{\sqrt{3}}{2}\cosh 1,\ 1,\ \tfrac{1}{2}\cosh 1\right) + s\left(\tfrac{\sqrt{3}}{2}\sinh 1,\ 1,\ \tfrac{1}{2}\sinh 1\right) + t\left(\tfrac{-1}{2}\cosh 1,\ 0,\ \tfrac{\sqrt{3}}{2}\cosh 1\right)$$

(5) [8 pts] Let $\mathbf{F}$ be the vector field on $\mathbf{R}^3$ given by $\mathbf{F} = xy\mathbf{i} + yz\mathbf{j} + xz\mathbf{k}$. About what axis is the circulation of $\mathbf{F}$ the greatest at the point $(2, 1, 3)$?

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & yz & xz \end{vmatrix} = (-y, -z, -x)$$

$$= (-1, -3, -2) \text{ at } (2,1,3)$$

~~Circle~~

Circulation is greatest about axis give by $\nabla \times F$ at the point. So axis is $(-1, -3, -2)$

(6) [12 pts] (a) Give a careful statement of the Divergence Theorem.

Let V be a solid region in $\mathbb{R}^3$. Let $\vec{F}$ be a VF on V. Let $S = \partial V$ be boundary surface of V oriented outwards. Then

$$\iiint_V (\nabla \cdot \vec{F})\, dV = \iint_{\partial V} \vec{F} \cdot d\vec{A}.$$

(b) Let $\mathbf{F}$ be a vector field on $\mathbf{R}^3$. Let $p \in \mathbf{R}^3$ and let B_ϵ be the ball of radius $\epsilon > 0$ centered at p. (So B_ϵ is a three-dimensional solid.) Let ∂B_ϵ denote the boundary surface of B_ϵ, oriented using the outward pointing normal vector field. Prove that

$$(\nabla \cdot \mathbf{F})(p) \;=\; \lim_{\epsilon \to 0} \frac{1}{\mathrm{Vol}(B_\epsilon)} \iint_{\partial B_\epsilon} \mathbf{F} \cdot d\mathbf{A}.$$

$$(\nabla \cdot \vec{F})(p) = \lim_{\varepsilon \to 0} \frac{\iiint_{B_\varepsilon} (\nabla \cdot \vec{F})\, dV}{\mathrm{Vol}(B_\varepsilon)}$$

by MVT Integrals

$$= \lim_{\varepsilon \to 0} \frac{\iint_{\partial B_\varepsilon} \vec{F} \cdot d\vec{A}}{\mathrm{Vol}(B_\varepsilon)}$$

(c) IF $\vec{F} = \rho \vec{v}$ THEN units of RHS are $\frac{1}{m^3} \cdot \frac{kg}{m^3} \cdot \frac{m}{s}\, m^2 = \frac{kg}{m^3 s}$

So $\nabla \cdot \vec{F}(p)$ = Rate of ~~change~~ decrease of fluid density at p

$= -\frac{\partial \rho}{\partial t}$

Pledge: *I have neither given nor received aid on this exam*

Signature: ______________________________

as $\nabla \cdot F(p) > 0$ means fluid