Natural and Step Responses of RLC Circuits

EE3301

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Learning Objectives

1. Be able to determine the natural responses of parallel and series RLC circuits
2. Be able to determine the step responses of parallel and series RLC circuits
3. Be able to determine the responses (both natural and transient) of second order circuits with op amps
Parallel RLC circuit

\[ i_C \quad + \quad i_L \quad + \quad i_R \]

\[ C \quad V_0 \quad L \quad R \]

\[ I_0 \]

\[ + \quad v \quad - \]

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The step response of a parallel $RLC$ circuit.
A series $RLC$ circuit.
The step response of a series $RLC$ circuit.
The Natural Response of a Parallel RLC

1. Using KCL,

\[
v(t)/R + \frac{1}{L} \int_0^t v(s) \, ds + C \frac{dv(t)}{dt} + I_0 = 0
\]

\[
\frac{d^2 v(t)}{dt^2} + \frac{1}{RC} \frac{dv(t)}{dt} + \frac{v(t)}{LC} = 0
\]
The Natural Response of a Series RLC

1. Using KVL,

\[ Ri(t) + \frac{1}{C} \int_0^t i(s) \, ds + L \frac{di(t)}{dt} + V_0 = 0 \]

\[ \frac{d^2 i(t)}{dt^2} + \frac{R}{L} \frac{di(t)}{dt} + \frac{i(t)}{LC} = 0 \]
The Natural Response of a Parallel/Series RLC

\[ v(t) = Ae^{st} \iff \text{parallel} \]

\[ s^2 + \frac{1}{RC} s + \frac{1}{LC} = 0 \Rightarrow \]

\[ s = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}} \]

---

\[ i(t) = Ae^{st} \iff \text{series} \]

\[ s^2 + \frac{R}{L} s + \frac{1}{LC} = 0 \Rightarrow \]

\[ s = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} \]
The Natural Response of a Parallel/Series RLC

There are 3 distinct cases.

Let

Then,

\[ \alpha = \frac{1}{2RC} \text{ or } \alpha = \frac{R}{2L} \]

\[ \omega_0 = \frac{1}{\sqrt{LC}} \]

\[ s = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} \]

1) \( \alpha = \omega_0 \) \hspace{0.5cm} \text{Critically-damped}

2) \( \alpha < \omega_0 \) \hspace{0.5cm} \text{Under-damped}

3) \( \alpha > \omega_0 \) \hspace{0.5cm} \text{Over-damped}
The Natural Response of a Parallel RLC

There are 3 distinct cases.

1) $\alpha = \omega_0 \Rightarrow s_1 = s_2 = -\alpha$
   - Critically-damped

2) $\alpha < \omega_0 \Rightarrow$
   \[
   \begin{align*}
   s_1 &= -\alpha + j \sqrt{\omega_0^2 - \alpha^2} \\
   s_1 &= -\alpha - j \sqrt{\omega_0^2 - \alpha^2}
   \end{align*}
   - Under-damped

3) $\alpha > \omega_0 \Rightarrow$
   \[
   \begin{align*}
   s_1 &= -\alpha + \sqrt{\alpha^2 - \omega_0^2} \\
   s_1 &= -\alpha - \sqrt{\alpha^2 - \omega_0^2}
   \end{align*}
   - Over-damped
The Natural Response of a Parallel/Series RLC

How does the response look like?

\[ v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \leftarrow parallel \]
\[ i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \leftarrow series \]

Initial conditions must be used to evaluate
How do we solve for the unknowns for over-damped (parallel)?

\[ v(0^+) = A_1 + A_2 \]

\[ \frac{dv(0^+)}{dt} = A_1 s_1 + A_2 s_2 \Rightarrow \]

\[ \frac{dv(0^+)}{dt} = \frac{i_c(0^+)}{C} = \frac{1}{C} \begin{bmatrix} -\frac{v(0^+)}{R} & -I_0 \end{bmatrix} \]

Initial voltage across the capacitor

Initial current through the inductor

Two equations with two unknowns
How do we solve for the unknowns for the over-damped case (series)?

\[ i(0^+) = A_1 + A_2 \]

\[ \frac{di(0^+)}{dt} = A_1 s_1 + A_2 s_2 \implies \]

\[ \frac{di(0^+)}{dt} = \frac{v_L(0^+)}{L} = \frac{1}{L} \left[ -Ri(0^+) - V_0 \right] \]

Initial current through the inductor

Initial voltage across the capacitor
Example 8.9

The graph shows the current $i_L$ (in mA) plotted against time $t$ (in $\mu$s). The three curves represent different damping conditions:

- **Underdamped** ($R = 625 \, \Omega$)
- **Overdamped** ($R = 400 \, \Omega$)
- **Critically damped** ($R = 500 \, \Omega$)
The circuit for Example 8.2.

\[ I_0 = 30 \text{mA}; \]

\[ v(0^+) = 12 \]
Figure 8.7  Example 8.2.

\[ s_1 = -5000 \text{; rad/} \text{sec} \]
\[ s_2 = -20,000 \text{; rad/} \text{sec} \]
The Natural Response of an under-damped Parallel/Series RLC

How does the response look like?

\[ v(t) = B_1 e^{-\alpha t} \cos(\omega_d t) + B_2 e^{-\alpha t} \sin(\omega_d t) \iff \text{parallel} \]
\[ i(t) = B_1 e^{-\alpha t} \cos(\omega_d t) + B_2 e^{-\alpha t} \sin(\omega_d t) \iff \text{series} \]

\[ \omega_d = \sqrt{\omega_0^2 - \alpha^2} \]

Initial conditions must be used to evaluate
How do we solve for the unknowns for under-damped (parallel)?

\[ v(0^+) = B_1 \]

\[ \frac{dv(0^+)}{dt} = \omega_d B_2 - \alpha B_1 \Rightarrow \]

\[ \frac{dv(0^+)}{dt} = \frac{i_c(0^+)}{C} = \frac{1}{C} \begin{bmatrix} -v(0^+) \\ -\frac{v(0^+)}{R} - I_0 \end{bmatrix} \]

Initial voltage across the capacitor

Initial current through the inductor
How do we solve for the unknowns for under-damped (series)?

\[
\begin{align*}
    i(0^+) &= B_1 \\
    \frac{di(0^+)}{dt} &= \omega_d B_2 - \alpha B_1 \Rightarrow \\
    \frac{di(0^+)}{dt} &= v_L(0^+) = \frac{1}{L} \left[ -Ri(0^+) - V_0 \right]
\end{align*}
\]

Initial current through the inductor

Initial voltage across the capacitor
The circuit for Example 8.4

\[ I_0 = -12.25 \text{ mA}; \]

\[ v(0^+) = 0 \]
The voltage response for Example 8.4

\[ s_1 = -200 + j979.80 \]
\[ s_2 = -200 - j979.80 \]
The Natural Response of a Critically Damped Parallel/Series RLC

How does the response look like? \[ s_1 = s_2 = s \]

\[ v(t) = D_1 t e^{st} + D_2 e^{st} \]

Initial conditions must be used to evaluate.
How do we solve for the unknowns for critically-damped (parallel)?

\[ v(0^+) = D_2 \]

\[ \frac{dv(0^+)}{dt} = D_1 - \alpha D_2 \Rightarrow \]

\[ \frac{dv(0^+)}{dt} = \frac{i_c(0^+)}{C} = \frac{1}{C} \left[ -\frac{v(0^+)}{R} - I_0 \right] \]

- Initial voltage across the capacitor
- Initial current through the inductor
How do we solve for the unknowns for the critically-damped (series)?

\[ i(0^+) = D_2 \]

\[ \frac{di(0^+)}{dt} = D_1 - \alpha D_2 \Rightarrow \]

\[ \frac{di(0^+)}{dt} = \frac{v_L(0^+)}{L} = \frac{1}{L} \left[ -Ri(0^+) - V_0 \right] \]

Initial current through the inductor

Initial voltage across the capacitor
Example 8.5 (critically-damped)-R has been changed to make this happen

\[ \omega_0 = \sqrt{10^6} \]

\[ \alpha = \omega_0 \Rightarrow R = 4k\Omega \]

\[ s = -1000 \]

\[ v(t) = 98000te^{-1000t} \]
A circuit used to describe the step response of a parallel RLC circuit
The Step Response of a Parallel RLC

1. Using KCL,

\[ \frac{v_L(t)}{R} + i_L(t) + C\frac{dv_L(t)}{dt} = I \]
\[ \frac{L}{R} \frac{di_L(t)}{dt} + i_L(t) + LC\frac{d^2i_L(t)}{dt^2} = I \]
\[ \frac{d^2i_L(t)}{dt^2} + \frac{1}{RC} \frac{di_L(t)}{dt} + \frac{1}{LC} i_L(t) = \frac{I}{LC} \]
The Step Response of a Parallel RLC
(direct method)

1. First find the natural response
2. Add to the natural response the final value
3. Use the initial conditions to solve for coefficients

\[
\frac{d^2 i_{L,n}(t)}{dt^2} + \frac{1}{RC} \frac{di_{L,n}(t)}{dt} + \frac{1}{LC} i_{L,n}(t) = 0
\]

\[
i_L(t) = I_f + i_{L,n}(t)
\]
The Step Response of a Parallel RLC (direct method)

\[ i_L(0^+) \frac{di_L(0^+)}{dt} \leftarrow \text{known} \]

\[ i_L(t) = I_f' + A_1' e^{s_1 t} + A_2' e^{s_2 t} \]

\[ i_L(t) = I_f + B_1' e^{-\alpha t} \cos(\omega_d t) + B_2' e^{-\alpha t} \sin(\omega_d t) \]

\[ i_L(t) = I_f + D_1't e^{-\alpha t} + D_2' e^{-\alpha t} \]
A circuit used to illustrate the step response of a series RLC circuit.
The Step Response of a Series RLC

1. Using KVL,

\[ RC \frac{dv_c(t)}{dt} + v_c(t) + LC \frac{d^2v_c(t)}{dt^2} = V \]

\[ L \frac{di_L(t)}{dt} + Ri_L(t) + \frac{1}{C} \int^t i_L(t')dt' = V \]

\[ \frac{d^2v_c(t)}{dt^2} + \frac{R}{L} \frac{dv_c(t)}{dt} + \frac{1}{LC} v_c(t) = \frac{V}{LC} \]
The Step Response of a Series RLC (direct method)

\[ v_c(0^+) \frac{dv_c(0^+)}{dt} \equiv \text{known} \]

\[ v_c(t) = V_f + A_1 e^{s_1 t} + A_2 e^{s_2 t} \]

\[ v_c(t) = V_f + B_1 e^{-\alpha t} \cos(\omega_d t) + B_2 e^{-\alpha t} \sin(\omega_d t) \]

\[ v_c(t) = V_f + D_1 t e^{-\alpha t} + D_2 e^{-\alpha t} \]
The circuit for Example 8.12

\[ v_c(0^+) = 0, \quad \frac{dv_c(0^+)}{dt} = 0 \]

\[ v_c(t) = 48 + B_1 e^{-\alpha t} \cos(\omega_d t) + B_2 e^{-\alpha t} \sin(\omega_d t) \]

\[ \alpha = -1400; \quad \omega_d = 4800 \]
Second order circuits with op amps

\[ \frac{d^2 v_o(t)}{dt^2} = \frac{1}{R_1 C_1} \frac{1}{R_2 C_2} v_g(t) \]
Second order circuits with op amps

- This is a variation of the second order system
- The output is the double integration of the input
- Depending on the initial charges on the capacitors, the response will vary
- For a constant input, the output will increase indefinitely

\[
\frac{d^2 v_0(t)}{dt^2} = \frac{1}{R_1 C_1} \frac{1}{R_2 C_2} v_g(t)
\]

\[
v_g(t) = V_0
\]

\[
v_0(t) = \frac{V_0}{2R_1 C_1 R_2 C_2} t^2
\]
Second order circuits with op amps-imperfect integrator
\[ v_{01}(t) = -R_b C_2 \frac{dv_0(t)}{dt} - \frac{R_b C_2}{\tau_2} v_0(t) \]

\[ v_g(t) = -R_a C_1 \frac{dv_{01}(t)}{dt} - \frac{R_a C_1}{\tau_1} v_{01}(t) \]

\[ \tau_1 = R_1 C_1; \tau_2 = R_2 C_2 \]

\[ \frac{d^2 v_0(t)}{dt^2} + \left( \frac{1}{\tau_1} + \frac{1}{\tau_2} \right) \frac{dv_0(t)}{dt} + \left( \frac{1}{\tau_1 \tau_2} \right) v_0(t) = \frac{v_g(t)}{R_a C_1 R_b C_2} \]

\[ s_1 = \frac{-1}{\tau_1}; s_2 = \frac{-1}{\tau_2} \]

\[ v_g(t) = V_0 U(t) \]

\[ v_0(t) = V_f + A_1 e^{s_1 t} + A_2 e^{s_2 t} \Leftrightarrow v_0(0^+) = \frac{dv_0(0^+)}{dt} = 0 \]

\[ V_f = \frac{V_0 \tau_1 \tau_2}{R_a C_1 R_b C_2} = \frac{R_1 R_2 V_0}{R_a R_b} \]

If it is less than the supply voltage