Circuit Theory I

First order RL and RC circuits
Learning Objectives

1. Be able to determine the natural response of both \textit{RL} and \textit{RC} circuits.
2. Be able to determine the step response of both \textit{RL} and \textit{RC} circuits.
3. Know how to analyze circuits with sequential switching.
4. Be able to analyze op amp circuits containing resistors and a single capacitor.
Circuits containing energy-storage elements

- The simplest circuits containing energy storage elements are those consisting of a “single” energy-storage element embedded in a linear network of sources and resistors.
- However complex the linear network of sources and resistors may be, we can always replace it with its Thevenin or Norton equivalent.

![Diagram of circuits containing energy-storage elements]
Circuits containing energy-storage elements

- Circuits that contain only one energy-storage element (or that contain multiple energy storage elements but in such a way that they can be reduced to a single equivalent element via series-parallel combinations) embedded in a linear network of sources and resistors can always be reduced to one of the following forms:
First order RC and RL circuits

- We wish to find the voltage and current developed by the energy storage element. The manner in which voltage or current varies with time is referred as **time response**

\[
RC \frac{dv(t)}{dt} + v(t) = v_s(t)
\]

\[
\frac{L}{R} \frac{di(t)}{dt} + i(t) = i_s(t)
\]

- Since the voltages and currents of the “basic” RL and RC circuits are described by first order differential equations the basic RL and RC circuits are a.k.a. **first order circuits**
First Order RC and RL circuits

Both equations have the same structure:

\[
\tau \frac{dy(t)}{dt} + y(t) = x(t)
\]

\[RC \frac{dv(t)}{dt} + v(t) = v_s(t)\]

\[\frac{L}{R} \frac{di(t)}{dt} + i(t) = i_s(t)\]

\[y(t) = \text{unknown variable} = \begin{cases} v(t) & \text{for the capacitive case} \\ i(t) & \text{for the inductive case} \end{cases}\]

\[x(t) = \text{forcing function} = \begin{cases} v_s(t) & \text{for the capacitive case} \\ i_s(t) & \text{for the inductive case} \end{cases}\]

\[\tau = \text{time constant} = \begin{cases} RC & \text{for the capacitive case} \\ L/R & \text{for the inductive case} \end{cases}\]

Both RC and L/R have the dimensions of time
Analysis of basic RL and RC circuits

- **Goal:** Determine the currents and voltages that arise when energy is either acquired or released by an inductor or capacitor in response to a change in a voltage or current source.

- **NOTE:** A circuit containing multiple capacitances (or inductances) is still a first order circuit if its topology allows for the capacitances (or inductances) to be reduced to a single equivalent capacitance (or inductance) through repeated usage of parallel and series combinations.
Response of basic RL and RC circuits

• Since the circuits we are dealing with are LTI, we can apply superposition to solve them.

\[
\tau \frac{dy(t)}{dt} + y(t) = x(t)
\]

• As a result of applying superposition we can expect the solution to be formed by two components: 
  \[y(t) = y_*(t) + y_{**}(t)\]
  – One component that does not depend on the driving source
    \[
    \tau \frac{dy_*(t)}{dt} + y_*(t) = 0
    \]
  – One component that depends on the particular driving source
    \[
    \tau \frac{dy_{**}(t)}{dt} + y_{**}(t) = x(t)
    \]
Response of basic RL and RC circuits

- At a first glance it may seem odd to expect the circuit to produce a non-zero response with a zero forcing source, however this behavior stems from the ability of capacitors and inductors to store energy.
- It is precisely this energy that allows the circuit to sustain non-zero voltages and currents even in absence of any forcing source.
- This voltages and currents will persists until all of the initial energy has been used up by the resistance in the circuit.

\[
\tau \frac{dy_*(t)}{dt} + y_*(t) = 0
\]
Response of basic RL and RC circuits

- The component of the solution that does not depend on the driving (= forcing = excitation) source is called the **homogenous associated solution** (it is the solution of the homogeneous differential equation associated with the original differential equation) **OR** the **source free solution OR** the **natural response** (it depends only on the internal energy storage properties of the circuit not on its external sources)

\[ y_*(t) \equiv y_{HA}(t) \equiv y_N(t) \]

\[ \tau \frac{dy(t)}{dt} + y(t) = x(t) \quad \text{Original differential equation} \]

\[ \tau \frac{dy(t)}{dt} + y(t) = 0 \quad \text{Homogeneous (set the forcing function in the original differential equation to 0) differential equation associated} \]
Response of basic RL and RC circuits

- The component of the solution that depends on the particular driving source is called the **particular solution OR** the **forced response**

\[ y_{\text{sp}}(t) \equiv y_p(t) \equiv y_f(t) \]

- In summary, the **total response** (= general solution) of a basic RL or RC circuit has the following structure:

\[ y(t) = y_{HA}(t) + y_p(t) \equiv y_N(t) + y_f(t) \]
Natural response of basic RL circuit

• Let’s assume that the independent current source $i_s$ generates a constant current $I_0$ and that the switch has been in a closed position for a long long time (let’s say it has been closed since $t=-\infty$)

• all currents and voltages have constant values

• Prior the switch being opened the inductor appears as a short circuit:

$$v = L \frac{di_L}{dt} = L \frac{d(\text{constant})}{dt} = 0$$
Natural Response of basic RL circuit

- Because prior the switch being opened the inductor appears as a short circuit (the voltage across the inductance in zero) and there can be no current in either \( R_0 \) or \( R \). Therefore, all source current \( i_s = I_0 \) appears in the inductance.
- After the switch has been opened (that is after the source has been disconnected) the inductor begin releasing the stored energy.

\[
L \frac{di(t)}{dt} + Ri(t) = 0 \quad \text{for} \quad t \geq 0
\]
Natural Response of basic RL circuit

\[ L \frac{di(t)}{dt} + Ri(t) = 0 \iff L \frac{di}{dt} dt + R \cdot i \cdot dt = 0 \iff \]

\[ \iff L di = -R \cdot i \cdot dt \iff \frac{di}{i} = -\frac{R}{L} dt \]

\[ \int_{i(0)}^{i(t)} \frac{di}{i} = -\frac{R}{L} \int_{0}^{t} dt \iff \ln \left( \frac{i(t)}{i(0)} \right) = -\frac{R}{L} t \iff i(t) = i(0) e^{-(R/L)t} \quad \text{with} \quad t \geq 0 \]

Since an instantaneous change of current cannot occur in an inductor (unless the voltage across it is infinite) in the first instant after the switch has been opened, the current in the inductor remains unchanged:

\[ i(0) = i(0-) = I_0 \]

\[ i(t) = I_0 \cdot e^{-(R/L)t} \quad \text{with} \quad t \geq 0 \]

Exponential decay
Natural Response of basic RL circuit

\[ i(t) = I_0 \cdot e^{(R/L)t} \quad \text{with} \quad t \geq 0 \]

\[ v(t) = R \cdot i(t) = RI_0 \cdot e^{-(R/L)t} \quad \text{with} \quad t \geq 0 \]

At \( t=0 \) there is a step in the voltage
Natural Response of basic RL circuit

\[ v(t) = R \cdot i(t) = R I_0 \cdot e^{-(R/L)t} \quad \text{with} \quad t \geq 0 \]

At \( t=0 \) there is a step in the voltage

For \( t<0 \):

\[
v(0-) = L \left. \frac{dt}{dt} \right| \frac{dt}{t=0} = -L \frac{dI_0}{dt} = 0 \quad \text{with} \quad t < 0
\]

For \( t>0 \):

\[
v(0+) = -L \left. \frac{d}{dt} \left( I_0 \cdot e^{-(R/L)t} \right) \right|_{t=0} = RI_0 \quad \text{with} \quad t \geq 0
\]

Be careful before opening the switch the current in \( L \) and \( R \) are different so at \( t=0^- \) the currents are: \( i_L=-I_0 \) and \( i_R=i=0 \)

After opening the switch the current in \( L \) and \( R \) are the same: \( i_L=i_R=i \) for \( t \geq 0 \)

\[
v(0-) = R \cdot i_R(0-) = R \cdot 0 = 0
\]

\[
v(0+) = R \cdot i_R(0+) = RI_0
\]
Natural Response of basic RL circuit

- The energy delivered to the resistor during any interval of time after the switch has been opened is:

\[ p(t) = i(t)v(t) = I_0 \cdot e^{(-R/L)t} \cdot I_0 R \cdot e^{(-R/L)t} = I_0^2 R \cdot e^{-(R/L)t} \quad \text{for} \quad t \geq 0 \]

\[ w(t) = \int_0^t p(t) \, dt = I_0^2 R \cdot \int_0^t e^{(-2R/L)t} \, dt = I_0^2 R \cdot \left[ \frac{e^{(-2R/L)t}}{-2R/L} \right]_0^t = \frac{I_0^2 R}{-2R/L} \cdot \left[ e^{-(2R/L)t} \right]_0^t = \]

\[ = \frac{1}{2} LI_0^2 \left(1 - e^{-(2R/L)t}\right) \quad \text{for} \quad t \geq 0 \]

As \( t \to \infty \) the energy dissipated by the resistor approaches the initial energy stored in the inductor.
Natural response of basic RL circuit

**The time constant \( \tau \)**

The *time constant* \( \tau = \frac{L}{R} \) determines the rate at which the current or voltage decays (= approaches zero)

\[
\frac{di}{dt}\bigg|_{t=0} = \frac{d}{dt}(I_0 e^{-\left(\frac{R}{L}\right)t})\bigg|_{t=0} = -I_0 \frac{R}{L} = -\frac{I_0}{\tau}
\]

Graphical interpretation of the tangent: \( i - i_0 \approx m (t-t_0) \)

\[
i - I_0 = -\frac{I_0}{\tau} t
\]
Natural response of basic RL circuit

- **The time constant \( \tau \)**

\[
i(t) = I_0 e^{-t/\tau} \quad \text{with} \quad t \geq 0
\]

One \( \tau = L/R \) after the inductor has begun to release its stored energy to the resistor the current is reduced to \( e^{-1} (=0.37) \) of its initial value or in other words the response has decayed 63% (=1−0.37) of its entire decay

5\( \tau \) after the inductor has begun to release its stored energy the current become less than 1% of its initial value, thus for most practical purposes we consider the currents and voltages as having reached their final values
Natural response of basic RC circuit

- Let’s assume that the independent voltage source $v_s$ generates a constant voltage $V_0$ and that the switch has been in the “a” position for a long long time (let’s say it has been closed since $t=-\infty$)

- all voltages and currents have constant values

- Prior putting the switch in position b, the capacitor behaves as an open circuit:

$$i = C \frac{dv_c}{dt} = C \frac{d(\text{constant})}{dt} = 0$$
Natural response of basic RC circuit

- Since the voltage source cannot sustain a current, all the source voltage appear across the capacitor terminals:

  \[ v(0-) = V_0 \]

- After the switch has been moved to “b” (that is after the source has been disconnected) the capacitor begin releasing the stored energy

\[ C \frac{dv}{dt} + \frac{v}{R} = 0 \quad \text{for} \quad t \geq 0 \]
Natural response of basic RC circuit

• Since an instantaneous jump in the capacitor voltage would require an infinite spike of current, for finite current the voltage across the capacitor voltage must be continuous: $v(0-) = V_0 = v(0+)$.

$$C \frac{dv}{dt} + \frac{v}{R} = 0 \quad \text{for} \quad t \geq 0$$

$v(t) = v(0+)e^{-t/RC} = V_0e^{-t/RC} \quad \text{for} \quad t \geq 0$
Natural response of basic RC circuit

\[ v(t) = v(0+)e^{-t/RC} = V_0e^{-t/RC} \quad \text{for} \quad t \geq 0 \]

\[ i(t) = \frac{v(t)}{R} = \frac{V_0}{R} e^{-t/RC} \quad \text{for} \quad t \geq 0 \]

At \( t=0 \) there is a step in the current

\[ i(0-) = 0; \quad i(0+) = \frac{V_0}{R} \]
Natural Response of basic RC circuit

- The energy delivered to the resistor during any interval of time after the switch has been put on “b” is:

\[ p(t) = i(t)v(t) = \frac{V_0^2}{R} \cdot e^{-\frac{2t}{RC}} \text{ for } t \geq 0 \]

\[ w(t) = \int_0^t p(t)dt = \frac{1}{2}CV_0^2\left(1 - e^{-\frac{2t}{RC}}\right) \text{ for } t \geq 0 \]

As \( t \to \infty \) the energy dissipated by the resistor approaches the initial energy stored in the capacitor.
Natural response of basic RC circuit

- **The time constant** $\tau$

  \[ \tau = RC \]

  The time constant $= \tau = RC$
  Determines the rate at which the current or voltage decays ($= \text{approaches zero}$)

  \[ v(t) = V_0 e^{-t/\tau} \]

  slope of $v(t)$ at $t=0$: $-V_0/\tau$

  \[ v(t) = V_0 - \frac{V_0}{\tau} t \]

  \[ \text{One } \tau = RC \text{ after the capacitor has begun to release its stored energy to the resistor, the voltage is reduced to } e^{-1} (=0.37) \text{ of its initial value or in other words the response has decayed } 63\% (=1-0.37) \text{ of its entire decay} \]

  \[ 5\tau \text{ after the capacitor has begun to release its stored energy the voltage become less than } 1\% \text{ of its initial value, thus for most practical purposes we consider the currents and voltages as having reached their final values} \]
Properties of the Natural Response

\[ \tau \frac{dy(t)}{dt} + y(t) = 0 \]

In math this equation is called an **homogeneous first order differential equation with constant coefficients**, and its solution \( y(t) \) is referred as the homogeneous solution.

Physically, the solution is referred as the **source free response** or the **natural response**.

Lacking any forcing source the response of the circuit is driven solely by the initial energy of its energy storage elements.

\[ \tau \frac{dy(t)}{dt} + y(t) = 0 \iff \frac{dy(t)}{dt} = -\tau \cdot y(t) \]

Aside from the constant \(-\tau\) the unknown and its derivative must be of the same form.

Among the many functions encountered in calculus only the exponential function enjoys the unique property that its derivative it’s still exponential, so we expect the solution to be some exponential function:

\[ y(t) = A \cdot e^{st} \]
Properties of the Natural Response

• Plugging the expected solution back into the original equation:

\[ y(t) = Ae^{st} \]

\[ \tau \frac{dy(t)}{dt} + y(t) = 0 \iff (\tau s + 1) Ae^{st} = 0 \]

\[ \tau s + 1 = 0 \iff s = -\frac{1}{\tau} \]

\[ \text{Unless } A=0 \text{ which gives the trivial solution } y(t)=0 \]

\[ \tau s + 1 = 0 \]

This is usually called the characteristic equation

The root of the characteristic eq. has the dimension of reciprocal of time (=frequency) so it is called characteristic frequency or critical frequency or natural frequency

• At this point we found an expression for s but we still miss one for A. Finding out A is relatively easy, all we need to do is let \( t \to 0 \):

\[ y(t) = Ae^{st} \quad \underset{t \to 0}{\longrightarrow} \quad y(0) = A \]

\[ A \text{ is the initial condition } y(0) \text{ in the circuit (that is the initial voltage } v(0) \text{ across the capacitance or the initial current through the inductance)} \]
Properties of the Natural Response

- So “wrapping” it all together, the solution is:

$$y(t) = y(0)e^{-t/\tau}$$

$y(t)$ is an exponentially decaying function from the initial value $y(0)$ to the final value $y(\infty)=0$

Since the decay depends only on $y(0)$ and $\tau$ which are characteristic of the circuit irrespective of any particular driving source, this solution is called the natural response (or source free response or homogeneous solution)
Properties of the Natural Response

\[ y(\tau) = y(0)e^{-1} = \frac{1}{e} y(0) \approx 0.37 y(0) \]

\( \tau \) provides a measure of how rapidly the exponential decay

\( \tau \) represents the amount of time it takes for the natural response to decay to 37% of its initial value (that is, after a time \( \tau \) the response has decayed 63% (=100%–37%) of its entire decay)

An alternative way of looking at \( \tau \) is to say that it represents the instant at which the tangent to the natural response at the origin intercept the \( t \) axis

\[
\left. \frac{dy(t)}{dt} \right|_{t=\tau} = \left. \left[ \frac{1}{\tau} y(0)e^{-t/\tau} \right] \right|_{t=0} = -\frac{y(0)}{\tau} \text{ initial slope of the response curve}
\]
Properties of the Natural Response

The larger the value of $\tau$ the slower the rate of decay

Figure 7.16 The larger the time constant $\tau$, the slower the rate of decay of the natural response.

Although in theory the response reaches zero only in the limit $t \rightarrow \infty$, in practice it is common practice to regard the decay as essentially complete after $5\tau$ (since by this time the response has already dropped below 1% of its initial value, which is negligible in most cases of interests)
The s-plane

- It is useful to visualize the root of the characteristic equation as a point in a plane called the s-plane

- The further away the root from the origin the more rapid the exponential decay

- $\tau \text{ small } \rightarrow \text{ decay is fast}$
The s-plane

• It is interesting to note that in the limit of a root right at the origin (1/τ=0) we have τ=∞ which implies an infinitely slow decay. Physically, the condition τ=∞ can be achieved by letting R=∞ in the capacitive case, or R=0 in the inductive case.

• When open-circuited an ideal capacitor will retain its initial voltage indefinitely, so v(t)=v(0) for any t ≥ 0

• When short-circuited an ideal inductor will sustain its initial current indefinitely, so i(t)=i(0) for any t ≥ 0
Response to Arbitrary Forcing Functions

- Although finding the natural response of a circuit is definitely a good starting point to learn about the characteristic of a circuit, in general we need to be able to find the response of a circuit to any arbitrary forcing function \( x(t) \)

- It turns out that to study the response of a circuit to any arbitrary forcing function the special cases in which DC (= constant value sources) and AC (= sinusoidal sources) forcing functions are suddenly applied to the circuit are of special practical interest.
Response to Arbitrary Forcing Functions

\[ \tau \frac{dy(t)}{dt} + y(t) = x(t) \]

- An elegant way of solving the first order differential equation we are dealing with is to multiply both sides by:

\[ \frac{1}{\tau} e^{t/\tau} \]

\[ e^{t/\tau} \frac{dy(t)}{dt} + \frac{e^{t/\tau}}{\tau} y(t) = \frac{e^{t/\tau}}{\tau} x(t) \quad \iff \quad \frac{d}{dt} \left( e^{t/\tau} y(t) \right) = \frac{e^{t/\tau}}{\tau} x(t) \]

\[ \int_0^t \frac{d}{dt} \left( e^{t/\tau} y(t) \right) dt = \int_0^t e^{t/\tau} x(t) dt \]
Response to Arbitrary Forcing Functions

\[
\frac{d}{dt} \left( e^{t/\tau} y(t) \right) = \frac{e^{t/\tau}}{\tau} x(t) \quad \iff \quad \int_0^t \frac{d}{dt} \left( e^{t/\tau} y(t) \right) dt = \int_0^t \frac{e^{t/\tau}}{\tau} x(t) dt
\]

\[
\int_0^t d \left( e^{t/\tau} y(t) \right) = \frac{1}{\tau} \int_0^t x(t) e^{t/\tau} dt \quad \iff \quad e^{t/\tau} y(t) - e^0 y(0) = \frac{1}{\tau} \int_0^t x(t) e^{t/\tau} dt
\]

\[
y(t)e^{t/\tau} - y(0) = \frac{1}{\tau} \int_0^t x(t) e^{t/\tau} dt \quad \iff \quad y(t) = y(0)e^{-t/\tau} + \frac{1}{\tau} \int_0^t x(t) e^{t/\tau} dt
\]

In conclusion, the response consists of two components:

- The first component which is independent of the forcing function \( x(t) \) (this is the already familiar natural response)
- The second component which depends on the particular forcing function \( x(t) \) and is called the forced response
- The sum of the two components is called the complete response
Response to Arbitrary forcing function

\[ y_{\text{complete}} = y_{\text{natural}} + y_{\text{forced}} \]
\[ y_{\text{complete}} = y(t) \]
\[ y_{\text{natural}} = y(0)e^{-t/\tau} \]
\[ y_{\text{forced}} = \frac{1}{\tau} \int_{0}^{t} x(t)e^{t/\tau} \]

Depending on the form of the forcing function evaluating the integral analytically can be quite a challenge and numerical methods may have to be used instead.

The cases of greatest practical interest are two:
- the sudden application of a constant source (DC forcing function)
- the sudden application of a sinusoidal source (AC forcing function)
Response to the sudden application of a DC forcing function (= step function)

- A DC forcing function is a function of the type:
  \[ x(t) = X_s \quad \text{(with } X_s \text{ being a constant)} \]

- Assuming the DC forcing function is applied at \( t=0 \):
  \[ x(t) = \begin{cases} 
  0 & \text{for } t < 0 \\
  X_s & \text{for } t \geq 0 
  \end{cases} = X_s \cdot u(t) \quad \text{a.k.a. Step Function} \]
Step response of first order circuits

- We want to find the currents and voltages produced in first order RL or RC circuits when either a DC voltage source or a DC current source is **suddenly applied**

*Step Response of Basic RL Circuit*

\[ v_s(t) = V_s u(t) \]

\[ i_L(0-) = I_0 \]

\[ i_s(t) = \frac{V_s}{R} u(t) \]

\[ v(t) \]
Step response of first order circuits

*Step Response of Basic RC Circuit*

\[ v_c(0-) = V_0 \]

\[ v_s(t) = R I_s u(t) \]

\[ v_c(0-) = V_0 \]

\[ i_s(t) = I_s u(t) \]

Source Transformation
Step response of first order circuits

• We want to find the currents and voltages produced in first order circuits when a DC source is suddenly applied (that is the same as saying when a step source is applied)

\[ \tau \frac{dy(t)}{dt} + y(t) = X_s \quad \text{for} \quad t \geq 0 \]

\[ y_{\text{forced}} = \frac{1}{\tau} \int_0^t X_s e^{t/\tau} \, dt = \frac{X_s}{\tau} e^{-t/\tau} \int_0^t e^{t/\tau} \, dt = \frac{X_s}{\tau} e^{-t/\tau} \left[ \tau \cdot e^{t/\tau} \right]_0^t \quad \text{for} \quad t \geq 0 \]

\[ y_{\text{forced}} = X_s \left(1 - e^{-t/\tau}\right) \quad \text{for} \quad t \geq 0 \]

\[ y_{\text{natural}} = y(0)e^{-t/\tau} \quad \text{for} \quad t \geq 0 \]
Step response of first order circuits

• Thus the complete step response of a first order circuit is:

\[ y(t) = y_{\text{natural}} + y_{\text{forced}} = y(0)e^{-t/\tau} + X_S (1 - e^{-t/\tau}) = \]
\[ = y(0)e^{-t/\tau} + X_S - X_S e^{-t/\tau} \quad \text{for} \quad t \geq 0 \]

• Noting that for \( t \to \infty \) the previous equation yields:

\[ y(\infty) = y(0)e^{\infty} + X_S - X_S e^{\infty} = X_S = 0 \]

• We can also rewrite the complete step response as:

\[ y(t) = y(0)e^{-t/\tau} - X_S e^{-t/\tau} + X_S = [y(0) - y(\infty)]e^{-t/\tau} + y(\infty) \quad \text{for} \quad t \geq 0 \]

\[ \text{With } X_S = y(\infty) \]
Step response of first order circuits

\[ y(t) = [y(0) - y(\infty)]e^{-t/\tau} + y(\infty) \quad \text{for} \quad t \geq 0 \quad \text{(with} \quad y(\infty) = X_s) \]

- In summary, the complete step response of a first order circuit is an exponential “transient” from the initial value \( y(0) \) to the final value \( y(\infty) \). The exponential “transient” is again characterized by the time constant \( \tau \).
Step response of first order circuits

\[ y(t) = [y(0) - y(\infty)]e^{-t/\tau} + y(\infty) \quad \text{for} \quad t \geq 0 \]

- We can think of \( \tau \) as the time it takes for \( y(t) \) to accomplish 63% of the entire transition
- Alternatively, we can visualize \( \tau \) as the time at which the tangent to \( y(t) \) at the origin intercept the \( y(\infty) \) asymptote
Step response of first order circuits

- Writing the step response of a first order circuit as:
  \[ y(t) = [y(0) - y(\infty)]e^{-t/\tau} + y(\infty) \ 	ext{for} \ t \geq 0 \ 	ext{(with} \ y(\infty) = X_s) \]
  we can regard it as the sum of two components.

1. An exponential decaying component with initial magnitude \( y(0) - y(\infty) \) called the transient component

   \[ y_{\text{transient}}(t) = [y(0) - y(\infty)]e^{-t/\tau} \ 	ext{for} \ t \geq 0 \ 	ext{(with} \ y(\infty) = X_s) \]

2. A time independent component of value \( y(\infty) \) called the DC steady-state component, because this is the value to which the complete response will settle once the transient response has died out

   \[ y(t \rightarrow \infty) = [y(0) - y(\infty)]e^{-\infty/\tau} + y(\infty) \ 	ext{for} \ t \geq 0 \ 	ext{(with} \ y(\infty) = X_s) \]
Step response of first order circuits

\[ y(t) = y_{\text{transient}} + y_{\text{steady-state}} \quad \text{for} \quad t \geq 0 \]

\[ y_{\text{transient}} = [y(0) - y(\infty)]e^{-t/\tau} \quad \text{for} \quad t \geq 0 \]

\[ y_{\text{steady-state}} = y(\infty) \quad \text{for} \quad t \geq 0 \]

- It is interesting to note that \( y_{\text{transient}} \) has the same functional form as \( y_{\text{natural}} \) (and it consists of two terms \( y_{\text{natural}} \) itself \( y(0)e^{-t/\tau} \) and the term \( y(\infty) \) which is brought about by the forcing function):

\[ y_{\text{transient}} = y(0)e^{-t/\tau} - y(\infty)e^{-t/\tau} \quad \text{for} \quad t \geq 0 \]

- We also note that \( y_{\text{steady-state}} \) has the same form as the forcing function:

\[ y_{\text{steady-state}} = y(\infty) = X_s \]
Step response of basic RL circuit

\[ V_s = Ri(t) + L \frac{di(t)}{dt} \quad \text{for} \quad t \geq 0 \quad \Leftrightarrow \quad V_s u(t) = Ri(t) + L \frac{di(t)}{dt} \]

\[ i(t) = \frac{V_s}{R} + \left( I_0 - \frac{V_s}{R} \right) e^{-t/\tau} \quad \text{for} \quad t \geq 0 \]

Since the current through the inductor cannot change instantaneously unless we have an infinite voltage across it, we conclude that the current through the inductor before \((t=0^-)\) and after \((t=0)\) closing the switch must be the same \(i(0^-)=i(0)=I_0\).
Step response of basic RL circuit

\[ i(t) = \frac{V_s}{R} + \left( I_0 - \frac{V_s}{R} \right) e^{-t/\tau} \quad \text{for} \quad t \geq 0 \]
Step response of basic RL circuit

\[ i(t) = \frac{V_s}{R} + \left( I_0 - \frac{V_s}{R} \right) e^{-t/\tau} \quad \text{for} \quad t \geq 0 \]

\[ v(t) = L \frac{di(t)}{dt} = (V_s - I_0 R) e^{-t/\tau} \quad \text{for} \quad t \geq 0 \]

The voltage across L is zero before the switch closes, it jumps to \( V_s - I_0 R \) at the instant the switch closes (for \( t=0 \) the current though the circuit is \( i(0)=I_0 \) so the voltage across the resistor is \( I_0 R \)) and then it decays exponentially to zero as time goes by.
Step response of basic RC circuit

\[ C \frac{dv_c}{dt} + \frac{v_c}{R} = I_s \quad \text{for} \quad t \geq 0 \quad \Leftrightarrow \quad C \frac{dv_c}{dt} + \frac{v_c}{R} = I_s u(t) \]

\[ v_c(t) = I_s R + (V_0 - I_s R) e^{-t/\tau} \quad \text{for} \quad t \geq 0 \]

\[ i_c(t) = C \frac{dv_c}{dt} = (I_s - \frac{V_0}{R}) e^{-t/\tau} \quad \text{for} \quad t \geq 0 \]

Since the voltage across the capacitor cannot change instantaneously unless we have an infinite current through the capacitor, we conclude that the voltage across the capacitor before \((t=0^-)\) and after \((t=0)\) closing the switch must be the same \(v_c(0^-) - v_c(0) = V_0\)

The capacitor branch current changes instantaneously from 0 at \(t=0^-\) to \(I_s - V_0/R\) at \(t=0\)
Step response of basic RC circuit

\[ v_C(t) = I_S R + (V_0 - I_S R) e^{-t/\tau} \quad \text{for} \quad t \geq 0 \]

\[ i_C(t) = C \frac{dv_C}{dt} = \left(I_S - \frac{V_0}{R}\right) e^{-t/\tau} \quad \text{for} \quad t \geq 0 \]
Sequential Switching

- Whenever switching occurs more than once in a circuit we call it *sequential switching*.

- The key to solve sequential switching circuits is to find out the initial value at the times of switching for the capacitors and inductors in the circuit.

- At time of switching capacitive voltages (inductive currents) cannot change instantaneously unless infinite current (infinite voltage) can be sustained.
Example of Sequential Switching

- Find $i_L(t)$ for $0 \leq t < 35$ ms
- Find $i_L(t)$ for $t \geq 35$ ms
Example of Sequential Switching

- Find the voltage across the capacitor
- Plot the capacitor voltage versus time

\[-\infty < t < 0 \quad \text{switch in position a}\]
\[0 \leq t < 15 \text{ ms} \quad \text{switch in position b}\]
\[15 \text{ ms} \leq t < \infty \quad \text{switch in position c}\]
Unbounded (=Divergent) Response

- A circuit may grow rather than decay exponentially.
- This type of response called an unbound response is possible if the circuit contains dependent sources.
- In the case in which the circuit contains dependent sources the Thevenin’s equivalent resistance may be negative.
- A negative resistance generate a negative time constant so the resulting currents and voltages increases without limit (at least theoretically).

\[ v_o(t) = v_o(0^+) e^{-t/\tau} \]

**Critical Frequency:**

\[ s = -1/\tau \]

**Figure 8.34** A root on the right half of the horizontal axis of the s plane corresponds to an exponentially diverging natural response.
Unbounded (=Divergent) Response

• In practice, in an actual circuit, the response cannot grow indefinitely. The response eventually reaches a limiting value when a component breaks down or goes into saturation state (that is the component stops behaving linearly) prohibiting further increases in voltage and current.

\[ v_o(t) = v_o(0^+)e^{-t/\tau} \]

(\(\tau\) is negative)
Example of Unbounded Behavior

\[ R_{TH} = \frac{v_{aux}}{i_{aux}} \]

\[ i_{aux} = \frac{v_{aux}}{10K\Omega} - 7i_{\Delta} + i_{\Delta} \]

\[ i_{\Delta} = \frac{v_{aux}}{20K\Omega} \]

\[ i_{aux} = \frac{v_{aux}}{10K\Omega} - 6 \cdot \frac{v_{aux}}{20K\Omega} = v_{aux} (0.1 \times 10^{-3} - 0.3 \times 10^{-3}) \]

\[ \frac{v_{aux}}{i_{aux}} = -5K\Omega \]
First Order Op Amp Circuits

• Thanks to its ability to draw energy from its own power supply and inject it into the surrounding circuitry, the op amp can be used in ingenious ways to create effects that cannot be achieved with purely passive R,L,C components.

**Differentiating Amplifier**

**Integrating Amplifier**
**Differentiating Amplifier**

\[ i_c = i_R = i \]

\[ v_o(t) = -Ri(t) \]

\[ i(t) = C \frac{dv_I(t)}{dt} \]

\[ v_o(t) = -RC \frac{dv_I(t)}{dt} \]

- The output is proportional to the time derivative of the input.

- Unfortunately, unwanted noise is usually characterized by a rich content of high frequencies so using the differentiating amplifier has the undesired drawback of boosting unwanted noise.
Differentiator

Example

- During the time intervals over which $v_i$ increases, $i_C$ flows toward the right making $v_O$ negative.
- During the times when $v_i$ decreases, $i_C$ flows toward the left making $v_O$ positive.
Integrating Amplifier

\[ i_C = i_R \equiv i \]
\[ i(t) = \frac{v_I(t)}{R} \]
\[ i(t) = -C \frac{dv_O(t)}{dt} \]

\[ v_O(t) = -\frac{1}{RC} \int_0^t v_I(t) dt + v_O(0) \]

- The circuits produces an output voltage proportional to the time integral of the input voltage.
- \( v_O(0) \) is the output voltage at \( t=0 \). Its value is determined by the charge stored at this time in the capacitance.
Integrator

Example

- Of particular interest is the case where the input is a constant \( v_i(t) = V_i \)

\[
 v_o(t) = -\frac{V_i}{RC} t + v_o(0)
\]

- The rate at which the output ramps is

\[
 \frac{dv_o(t)}{dt} = -\frac{V_i}{RC}
\]

- With \( v_i > 0 \) we obtain a decreasing ramp, with \( v_i < 0 \) we obtain an increasing ramp

- When \( v_i > 0 \) then \( i_C \) flows toward the right and \( v_O \) decreases

- When \( v_i < 0 \) then \( i_C \) flows toward the left and \( v_O \) must increases

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Integrating Amplifier Example

- The switch remains in position (a) for 9ms and then it moves instantaneously to position (b)
- How long does it take for the op amp to saturate?
The integrating amplifier can perform the integration very well, but only within specified limits that avoid saturating the op amp.

The op amp saturates due to accumulation of charge on the feedback capacitor. We can prevent the op amp from saturating by placing a resistor in parallel with the feedback capacitor.

$$v_o(t) = -\frac{1}{RC} \int_0^t v_I(t) dt + v_o(0)$$
First Order Op Amp Circuits

- Theoretically we can design integrating and differential amplifier circuits by using an inductor instead of a capacitor.
- However fabricating capacitances for integrated circuits is much easier, so inductors are rarely used.