Bifurcation of limit cycles from a fold-fold singularity in planar switched systems

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1 Introduction

This paper investigates the existence of attractive limit cycles in switched systems of the form

$$\begin{aligned} \dot{x}(t) &= f^i(x(t), y(t)), \\ \dot{y}(t) &= g^i(x(t), y(t)), \end{aligned} \tag{1}$$

$$\begin{aligned} i &:= R, \quad \text{if } \ x(t) = x, \\ i &:= L, \quad \text{if } \ x(t) = -x, \end{aligned} \tag{2}$$

where f^L, f^R, g^L, g^R are smooth functions and $x \in \mathbb{R}$ is a parameter. To draw a trajectory of system (1)-(2) one not just needs the initial point $(x(0), y(0))$, but also the index of the system ($i = L$ or $i = R$) that governs the trajectory at $t = 0$. The trajectory is governed by the system i until it reaches one of the lines $\{-x\} \times \mathbb{R}$ or $\{x\} \times \mathbb{R}$, when i switches to $i = L$ or $i = R$ according to whether $\{-x\} \times \mathbb{R}$ or $\{x\} \times \mathbb{R}$ is hit. The trajectory then travels along system i until it reaches one of the switching lines again, when the same rule applies, see fig. 1. In this paper we only deal with solutions that intersect the switching lines transversally.

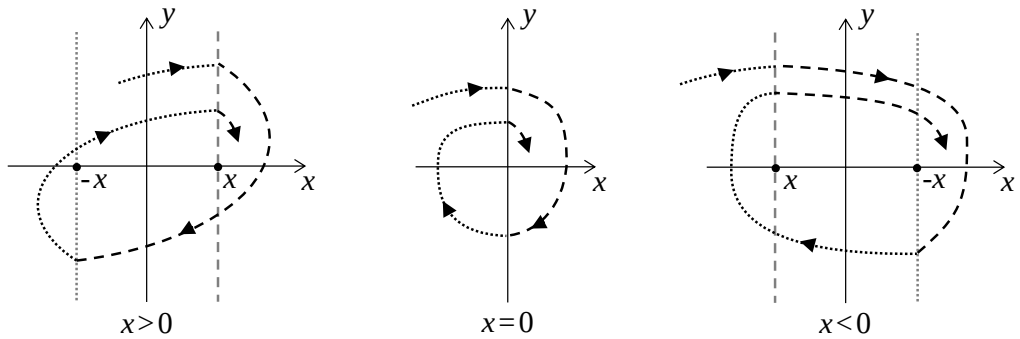

Figure 1: Sample trajectory of system (1)-(2) for different values of the parameter x. The dotted and dashed curves denote trajectories governed by system (1) with $i = L$ and $i = R$ respectively. The dotted line is where a switch to $i = L$ occurs. The dashed line is where a switch to $i = R$ occurs. The two lines coincide with $\{0\} \times \mathbb{R}$ when $x = 0$.

The conditions for the existence of limit cycles in linear systems (1)-(2) are proposed in Astrom [2] and Goncalves et al [4]. The case where (1)-(2) admits a describing function is addressed in Tsypkin [15]. These works found numerous applications in automatic control, where system (1)-(2) is equivalently formulated as

$$\dot{z} = h(z, u)$$

where z is a vector and u is a scalar control input that can only take a discrete set of values. The existence of a stable cycle for (1)-(2) with piecewise linear g^L and g^R is established in Andronov et al [1, Ch. III, §5] in the context of clock modeling. Stable limit cycles in nonlinear systems of form (1)-(2) are addressed in monographs by Barbashin [3] and Neimark [10] along with applications in electromechanical engineering. Other applications where a discrete-valued control is designed to produce limit cycles include switching convertors [6], intermittent therapy modeling in medicine [13], grazing management in ecology [9]. A particular model that motivated the current paper is an anti-lock braking system as introduced in [14, 11].

The paper deals with systems (1)-(2) whose limit cycle shrinks to a point $(0, y_0)$ when $x \to 0$. The latter can only happen when the vector fields (f^L, g^L) and (f^R, g^R) are oppositely directed, which is equivalent to saying

that $(0, y_0)$ is an invisible equilibrium (or switched equilibrium) of the reduced system

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \begin{cases} \begin{pmatrix} f^L(x(t), y(t)) \\ g^L(x(t), y(t)) \end{pmatrix}, & \text{if } x(t) < 0, \\ \begin{pmatrix} f^R(x(t), y(t)) \\ g^R(x(t), y(t)) \end{pmatrix}, & \text{if } x(t) > 0. \end{cases} \tag{3}$$

We consider $y_0 = 0$ to shorten notations. When the vectors $(f^L(0), g^L(0))$ and $(f^R(0), g^R(0))$ are transversal to the switching manifold $\{0\} \times \mathbb{R}$, the bifurcation of a limit cycle of (1)-(2) from an invisible equilibrium 0 of (3) is of interest in power electronics. This type of bifurcations is a subject of a different paper. In this paper we address the situation where both $(f^L(0), g^L(0))$ and $(f^R(0), g^R(0))$ are parallel to the line $\{0\} \times \mathbb{R}$, which takes places in anti-lock braking systems and switched mechanical oscillators (see Sec. 3). By the other words, we consider the case where (3) verifies

$$f^L(0) = f^R(0) = 0, \tag{4}$$

i.e. where the origin is a fold-fold singularity of (3). The main result of this paper is a sufficient condition that ensures bifurcation of a limit cycle of (1)-(2) from a fold-fold singularity of (3). This type of bifurcation is termed border-splitting bifurcation in [8].

System (1)-(2) can be viewed as a result of a discontinuous perturbation of the switching manifold in system (3). In this way, the current work complements the results by Guardia et al [5] and Kuznetsov et al [7] on bifurcations of limit cycles from a fold-fold singularity of (3) under smooth perturbations of the vector fields of (3). Relevant results for continuous systems (3) are obtained by Simpson-Meiss [12] and Zou-Kuepper-Beyn [17].

2 The main result

The main result of this paper is achieved by analyzing the normal form of the Poincare map $y \mapsto P(y)$ of system (1)-(2) induced by the line $\{x\} \times \mathbb{R}$. To construct the map P, we consider the flow of system (1) with $i = R$ from the line $\{x\} \times \mathbb{R}$ to the line line $\{-x\} \times \mathbb{R}$. The respective map is denoted by P^R and is called the point transformation. For example, $P^R(A) = B$ in the example of fig. 1. Furthermore, we denote by P^L the point transformation of (1) with $i = L$ from the line $\{-x\} \times \mathbb{R}$ to the line $\{x\} \times \mathbb{R}$. In particular, $P^L(B) = C$ for the flow of fig. 1. The Poincare map P is obtained as a composition

$$P = P^L \circ P^R,$$

i.e. $P(A) = C$ for the points from fig. 1. In section 2.1 we derive normal forms for the point transformations P^L and P^R, that will allow us do draw the required conclusions about P.

2.1 The normal form of a point transformation in the neighborhood of a fold-fold singularity

Consider a planar system

$$\begin{aligned} \dot{x} &= f(x, y), \\ \dot{y} &= g(x, y). \end{aligned} \tag{5}$$

Assuming that the origin is a fold-fold singularity of (5), we are going to spot the trajectories x of (5) that originate in $\{x\} \times \mathbb{R}$ at $t = 0$ and reach $\{-x\} \times \mathbb{R}$ at $t = T$ while crossing $\{x\} \times \mathbb{R}$ at just one point $t_* \in (0, T)$. This will allow as to view the point transformation of (5) from a suitable subset of $\{x\} \times \mathbb{R}$ to $\{-x\} \times \mathbb{R}$ as a composition $\widetilde{\mathcal{P}} \circ \mathcal{P}$, where $\mathcal{P}$ and $\widetilde{\mathcal{P}}$ are defined as $\mathcal{P}(x(0)) = x(T)$ and $\widetilde{\mathcal{P}}(x(t_*)) = x(T)$ respectively, see fig. 2.1.

The next lemma is saying that any solution of (5) that originates in $\{x\} \times \mathbb{R}$ reaches $\{x\} \times \mathbb{R}$ again in forward or backward time. This return is locally unique and the respective local map from $\{x\} \times \mathbb{R}$ to itself can be expanded as (9).

Lemma 1 *Assume that C^4 maps f and g satisfy*

$$f(0) = 0, \;\; f'_y(0)g(0) \neq 0. \tag{6}$$

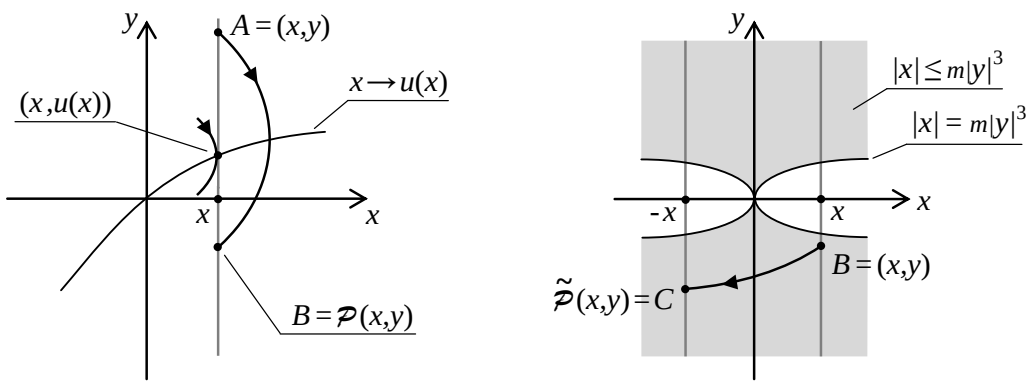

Figure 2: The two parts $A \mapsto B$ and $B \mapsto C$ of the point transformation $A \mapsto C$ of (5) induced by the lines $\{x\} \times \mathbb{R}$ and $\{-x\} \times \mathbb{R}$.

Let $u(x) \to 0$ as $x \to 0$ be the unique function such that $f(x, u(x)) = 0$ all $|x| \le \delta$, where $\delta > 0$ is a suitable constant. The constant $\delta > 0$ can be diminished so that, for all $y \ne u(x)$, $|x| \le \delta$, and $|y| \le \delta$ the equation

$$X(t, x, y) = x \tag{7}$$

admits a unique non-zero C^2 solution $T(x, y) \to 0$ as $y \to u(x)$. For this solution,

$$T'_y(x, u(x)) = -\frac{2}{g(x, u(x))}. \tag{8}$$

The C^2 map

$$\mathcal{P}(y) = Y(T(x, y), x, y)$$

expands as

$$\mathcal{P}(y) = -y + \alpha y^2 + \mathcal{R}(x, y), \tag{9}$$

where

$$\alpha = 2\frac{f'_x(0) + g'_y(0)}{g(0)} + \frac{f''_{yy}(0)}{f'_y(0)}, \qquad \mathcal{R}(0, 0) = \mathcal{R}'_y(0, 0) = \mathcal{R}''_{yy}(0, 0) = 0. \tag{10}$$

Proof. Step 1. *The existence of $T(x, y)$.* The existence of $u(t)$ under condition (6) follows from the Implicit Function Theorem (see [16, § 8.5.4, Theorem 1]). To solve (7) we expand X in Taylor series as

$$X(t, x, y) = x + X'_t(0, x, y)t + X''_{tt}(0, x, y)\frac{t^2}{2} + \Delta(t, x, y)t.$$

Equation (7) takes the form

$$X'_t(0, x, y) + X''_{tt}(0, x, y)\frac{t}{2} + \Delta(t, x, y) = 0, \tag{11}$$

where uniformly in $(x, y) \in [-\delta, \delta]^2$

$$\lim_{t\to 0} \Delta(t, x, y) = \lim_{t\to 0} \Delta'_t(t, x, y) = \lim_{t\to 0} \Delta'_y(t, x, y) = \lim_{t\to 0} \Delta''_{ty}(t, x, y) = \lim_{t\to 0} \Delta''_{yy}(t, x, y) = 0,$$
$$\lim_{t\to 0} \Delta''_{tt}(t, x, y) = \frac{x'''_{ttt}(0, 0, y)}{3}, \quad \Delta'_x(0, 0) = \Delta''_{tx}(0, 0) = \Delta''_{xx}(0) = 0.$$

By the Implicit Function Theorem there exists a unique $T(x, y) \to 0$ as $(x, y) \to 0$ that solves (7) in $[-\delta, \delta]^2$ and $T'_y(x, u(x))$ is given by (8). Since $T(x, u(x)) = 0$, then condition (6) implies that $T(x, y) \ne 0$ for all $y \in [-\delta, u(x)) \cup (u(x), \delta]$. Replacing t by $T(x, y)$ in (11) and taking the derivative with respect to x and y one can express $T'_x(x, y)$ and $T'_y(x, y)$ as

$$\begin{aligned} T'_x(x, y) &= \frac{X''_{tx}(0, x, y) + (1/2)X'''_{ttx}(0, x, y)t + \Delta'_x(T(x, y), x, y)}{-(1/2)x''_{tt}(0, x, y) - \Delta'_t(T(x, y), x, y)}, \\ T'_y(x, y) &= \frac{X''_{ty}(0, x, y) + (1/2)X'''_{tty}(0, x, y)t + \Delta'_y(T(x, y), x, y)}{-(1/2)x''_{tt}(0, x, y) - \Delta'_t(T(x, y), x, y)}, \end{aligned}$$

from where we conclude that T is C^2 with $T''_{yy}(0)$ given by

$$T''_{yy}(0) = 4\frac{f'_x(0) + g'_y(0)}{g(0)^2} + 2\frac{f''_{yy}(0)}{f'_y(0)g(0)}.$$

Step 2. *The expansion of* $\mathcal{P}(y)$. From $T(x, u(x)) = 0$ and $Y'_t(0, x, u(x))T'_y(x, u(x)) = -2$ one has $\mathcal{P}(u(x)) = u(x)$ and $\mathcal{P}'(u(x)) = -1$. Computing $\mathcal{P}''$ yields

$$\frac{1}{2}\mathcal{P}''(0) = \frac{1}{2}\left(Y''_{tt}(0)T'_y(0)^2 + Y'_t(0)T''_{yy}(0) + 2T''_{ty}(0)T'_y(0) + Y''_{yy}(0)\right) = \frac{1}{2}g(0)T''_{yy}(0) = \alpha_0. \tag{12}$$

Therefore, the Taylor series for $\mathcal{P}(y)$ around $y = u(x)$ is

$$\begin{aligned} \mathcal{P}(y) &= \mathcal{P}(u(x)) + \mathcal{P}'(u(x))(y - u(x)) + \mathcal{P}''(y_*)(y - u(x))^2 = \\ &= 2u(x) - y + \mathcal{P}''(y_*)(y - u(x)) \ =: \ -y + \alpha_0 y^2 + \mathcal{R}(x, y). \end{aligned} \tag{13}$$

The C^2 smoothness of $\mathcal{R}$ follows from the C^2 smoothness of $\mathcal{P}$, which is a consequence of C^2 smoothness of T.

The next lemma is saying that any solution of (5) that originates at (x, y) with $|x| \le m|y|^3$ reaches $\{-x\} \times \mathbb{R}$ in forward or backward time. This return is locally unique and the respective local map from $\{x\} \times \mathbb{R}$ to $\{-x\} \times \mathbb{R}$ can be expanded on $|x| \le m|y|^3$ as (16).

Lemma 2 *Assume that both f and g are C^4 and*

$$f(0) = 0, \ \ f'_y(0) \neq 0.$$

Then, for any $m > 0$ there exists $\delta > 0$ such that for all $m\sqrt[3]{|x|} \le |y|$ and $|y| \le \delta$ the equation

$$X(\widetilde{T}, x, y) = -x \tag{14}$$

admits a unique solution $|\widetilde{T}(x, y)| \le \dfrac{4}{m^3|f'_y(0)|}y^2$. This solution is C^2 in $0 < m\sqrt[3]{|x|} < |y| < \delta$ and

$$\widetilde{T}(x, y) \cdot \frac{y}{x} \to -\frac{2}{f'_y(0)} \quad \text{as} \quad |x| \le m|y|^3, \ y \to 0, \quad \text{where} \quad \beta = -\frac{2g(0)}{f'_y(0)}. \tag{15}$$

The respective map

$$\widetilde{\mathcal{P}}(y) = Y(\widetilde{T}(x, y), x, y)$$

expands as

$$\widetilde{\mathcal{P}}(y) = y + \beta\frac{x}{y} + \widetilde{R}(x, y), \tag{16}$$

where

$$\lim_{|x| \le m|y|^3, \ y \to 0} \frac{\widetilde{R}(x, y)}{y^2} = 0, \quad \lim_{|x| \le m|y|^3, \ y \to 0} \frac{\widetilde{R}'_y(x, y)}{y} = 0, \quad \lim_{|x| \le m|y|^3, \ y \to 0} \widetilde{R}'_x(x, y)y = 0. \tag{17}$$

Proof. *Step 1: The existence and uniqueness of* $\widetilde{T}$. Introduce

$$F(t, x, y) = \frac{X(t, x, y) + x}{y}.$$

Then $F(0, x, y) = 2x/y \le 2my^2$ when $|x| \le m|y|^3$ and $F'_t(t, x, y) \to X''_{ty}(0) = f'_y(0)$ as $|t| \le k|y|^2$, $|x| \le m|y|^3$, $y \to 0$. The latter holds for any fixed $k > 0$ and, in particular, for $k = 4m/f'_y(0)$. Thus, the existence, uniqueness, differentiability and the estimate for $\widetilde{T}$ follows from Theorem 2.

Step 2: The asymptotic of $\widetilde{T}(x, y)x/y$. Expanding $X(t, x, y) = x + X'_t(t_*, x, y)t$, we can rewrite (14) as

$$2x + X'_t(t_*, x, y)\widetilde{T}(x, y) = 0,$$

from where

$$\widetilde{T}(x,y)\cdot\frac{y}{x}=-\frac{2}{X''_{tt}(t_{**},x,y)\dfrac{t_*}{y}+\dfrac{X'_t(0,x,y)}{y}}.$$

Step 3: The asymptotic of $\widetilde{R}(x,y)y$. Expanding $X(t,x,y)=x+\bar{\Delta}(t,x,y)$, we can rewrite (14) as $2x+\bar{\Delta}(\widetilde{T}(x,y),x,y)=0$, from where

$$\widetilde{T}'_x(x,y)y=\frac{-2-\bar{\Delta}'_x(\widetilde{T}(x,y),x,y)}{\dfrac{\widetilde{\Delta}'_t(\widetilde{T}(x,y),x,y)}{y}}\to-\frac{2}{X''_{ty}(0)}\quad\text{as}\quad m\sqrt[3]{|x|}\le|y|,\ \ y\to 0.$$

Therefore,

$$\widetilde{R}'_x(x,y)y\to Y'_t(0)\left(-\frac{2}{X''_{ty}(0)}\right)-\beta=0\quad\text{as}\quad m\sqrt[3]{|x|}\le|y|,\ \ y\to 0.$$

Step 4: The expansion of $\widetilde{P}$. Expanding $X(t,x,y)=x+X'_t(0,x,y)t+\widetilde{\Delta}(t,x,y)$, we can rewrite (14) as

$$2x+X'_t(0,x,y)\widetilde{T}(x,y)+\widetilde{\Delta}(\widetilde{T}(x,y),x,y)=0, \tag{18}$$

from where

$$\widetilde{T}(x,y)=-\frac{2x}{X'_t(0,x,y)}-\frac{\widetilde{\Delta}(\widetilde{T}(x,y),x,y)}{X'_t(0,x,y)}.$$

Therefore, for

$$\widetilde{P}(x,y)=y+Y'_t(t_*,x,y)\widetilde{T}(x,y)=y-\frac{2Y'_t(0)}{X''_{ty}(0)}\cdot\frac{x}{y}+\widetilde{R}(x,y), \tag{19}$$

where

$$\widetilde{R}(x,y)=-\frac{2Y'_t(t_*,x,y)x}{X'_t(0,x,y)}-\frac{Y'_t(t_*,x,y)\widetilde{\Delta}(\widetilde{T}(x,y),x,y)}{X'_t(0,x,y)}+\frac{2Y'_t(0)}{X''_{ty}(0)}\cdot\frac{x}{y}.$$

$$\begin{aligned}\frac{\widetilde{R}(x,y)}{y^2}&=\left(-\frac{2Y'_t(t_*,x,y)}{\dfrac{X'_t(0,x,y)}{y}}+\frac{2Y'_t(0)}{X''_{ty}(0)}\right)\frac{x}{y^3}-\frac{Y'_t(t_*,x,y)}{\dfrac{X'_t(0,x,y)}{y}}\cdot\frac{\widetilde{\Delta}(\widetilde{T}(x,y),x,y)}{y^3},\\ \left|\frac{\widetilde{\Delta}(\widetilde{T}(x,y),x,y)}{y^3}\right|&=\left|\frac{X'_t(t_*,x,y)t-X'_t(0,x,y)t}{y^3}\right|=\left|\frac{X''_{tt}(t_{**},x,y)t_*t}{y^3}\right|\le|X''_{tt}(t_{**},x,y)|\cdot\frac{4m}{|f'_y(0)|}\cdot|y|.\end{aligned}$$

Differentiating (18) with respect to y,

$$\frac{\widetilde{T}'_y(x,y)}{y}=\frac{-X''_{ty}(0,x,y)\dfrac{\widetilde{T}(x,y)}{y^2}-\dfrac{\widetilde{\Delta}'_y(\widetilde{T}(x,y),x,y)}{y^2}}{\dfrac{X'_t(0,x,y)}{y}+\dfrac{\widetilde{\Delta}'_t(\widetilde{T}(x,y),x,y)}{y}}$$

Differentiating (19) with respect to y,

$$\begin{aligned}\frac{\widetilde{R}'_y(x,y)}{y}&=\frac{Y'_y(\widetilde{T}(x,y),x,y)-1}{y}+Y'_t(\widetilde{T}(x,y),x,y)\frac{\widetilde{T}'_y(x,y)}{y}-\frac{2Y'_t(0)}{X''_{ty}(0)}\cdot\frac{x}{y^3}=Y'''_{yy}(t_*,x,y)\frac{\widetilde{T}(x,y)}{y}+\\&+\left(Y'_t(\widetilde{T}(x,y),x,y)\left(\frac{X'_t(0,x,y)}{y}+\frac{\widetilde{\Delta}'_t(\widetilde{T}(x,y),x,y)}{y}\right)^{-1}\cdot\frac{2X''_{ty}(0,x,y)}{\dfrac{X'_t(0,x,y)}{y}}-\frac{2Y'_t(0)}{X''_{ty}(0)}\right)\frac{x}{y^3}+\\&+\left(\frac{X'_t(0,x,y)}{y}+\frac{\widetilde{\Delta}'_t(\widetilde{T}(x,y),x,y)}{y}\right)^{-1}\left(X''_{ty}(0,x,y)\frac{\widetilde{\Delta}(\widetilde{T}(x,y),x,y)}{y^3}\cdot\frac{1}{\dfrac{X'_t(0,x,y)}{y}}-\frac{\widetilde{\Delta}'_y(\widetilde{T}(x,y),x,y)}{y^2}\right)\end{aligned}$$

□

Lemma 2 provides information about those trajectories of (5) only whose initial conditions (x, y) satisfy $|x| \leq m|y|^3$. That is why we will need the following details about the map $\mathcal{P}$ when studying the composition $\widetilde{\mathcal{P}} \circ \mathcal{P}$.

Corollary 1 *Under the conditions of lemma 1, for any $m > 0$,*

$$\lim_{m\sqrt[3]{|x|}\leq|y|,\ y\to 0} \frac{T(x,y)}{y} = -\frac{2}{g(0)}, \qquad \lim_{m\sqrt[3]{|x|}\leq|y|,\ y\to 0} \frac{R(x,y)}{y^2} = 0, \qquad \lim_{m\sqrt[3]{|x|}\leq|y|,\ y\to 0} \frac{R'_y(x,y)}{y} = 0.$$

In particular, for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\mathcal{P}([m|x|^q, \delta]) \subset [-(1+\varepsilon)\delta, -(m-\varepsilon)|x|^q], \quad \mathcal{P}([-\delta, -m|x|^q]) \subset [(m-\varepsilon)|x|^q, (1+\varepsilon)\delta]$$

for all $|x| \leq \dfrac{1}{m^3}\delta^3$.

Proof.

$$\begin{aligned}
\frac{T(x,y)}{y} &= \frac{T'_x(x_*, y)x + T(0,y)}{y} \\
\frac{R'_y(x,y)}{y} &= \frac{R'_y(x,y) - R'_y(x,0) + R'_y(x,0)}{y} = \mathcal{R}''_{yy}(x, y_*) + \frac{\mathcal{R}''_{xy}(x_*, 0)x}{y} \\
\frac{R(x,y)}{y^2} &= \frac{R(x,y) - R(x,0) + R(x,0)}{y^2} = \frac{R'_y(x,y_*)}{y} + \frac{R(x,0)}{y^2} = \frac{R'_y(x,y_*)}{y} + \frac{R'_x(x_*,0)x}{y^2}.
\end{aligned}$$

Let us $\delta > 0$ be so small that $\dfrac{|\alpha y^2 + \mathcal{R}(x,y)|}{|y|} \leq \varepsilon$ for all $x \leq m|y|^3$ and $|y| \leq \delta$. For these values of x and y,

$$-y(1 - \varepsilon \cdot \mathrm{sign}|y|) \leq \mathcal{P}(y) \leq -y(1 - \varepsilon \cdot \mathrm{sign}|y|),$$

which implies the required inclusions for the values of $\mathcal{P}$. □

The next lemma simply computes the expansion of the composition of maps $\widetilde{\mathcal{P}}$ and $\mathcal{P}$ on $|x| \leq m|y|^3$.

Corollary 2 *Assume that the conditions of lemmas 1 and 2 hold and $\mathcal{P}$ and $\widetilde{\mathcal{P}}$ are the maps provided by these lemmas. Then*

$$\widetilde{\mathcal{P}}(\mathcal{P}(y)) = -y + \alpha y^2 - \beta\frac{x}{y} + r(x,y),$$

where, for any $m > 0$, the map Δ is C^2 in $0 < m\sqrt[3]{|x|} < |y|$ and

$$r(0,0) = \lim_{|x|\leq m|y|^3,\ y\to 0} \frac{r'_y(x,y)}{y} = \lim_{|x|\leq m|y|^3,\ y\to 0} r'_x(x,y)y = 0. \tag{20}$$

For any $m > 0$ there exists $\delta > 0$ such that, for $m\sqrt[3]{|x|} < \delta$ and

$$\mathrm{sign}(x) = -\mathrm{sign}(f'_y(0)g(0)) \tag{21}$$

the map $\widetilde{\mathcal{P}} \circ \mathcal{P}$ describes the transformation of the interval

$$\{x\} \times [-\mathrm{sign}\,(g(0))m\sqrt[3]{|x|}, -\mathrm{sign}\,(g(0))\delta] \tag{22}$$

of the line $\{x\} \times \mathbb{R}$ to the line $\{-x\} \times \mathbb{R}$ under the action of the flow of (5).

Proof. *Step 1: Properties of r.* Direct computation leads to

$$r(x,y) = \widetilde{\mathcal{R}}\left(x, -y + \alpha y^2 + \mathcal{R}(x,y)\right) + \mathcal{R}(x,y) + \widetilde{\mathcal{R}}(x,y) + \beta\frac{x}{y} + \beta\frac{x}{-y + \alpha y^2 + \mathcal{R}(x,y)}$$

and the required properties of r follows from the respective properties (10) and (17) of $\mathcal{R}$ and $\widetilde{\mathcal{R}}$.

Step 2: The relation between the map $\widetilde{\mathcal{P}} \circ \mathcal{P}$ and the flow of (5). The map $\widetilde{\mathcal{P}} \circ \mathcal{P}$ is a point transformation for the flow of (5), if the following two properties hold.

1) Both $T(x,y)$ and $\widetilde{T}(x,y)$ that appear in the definitions of $\widetilde{\mathcal{P}}$ and $\mathcal{P}$ in lemmas 1 and 2 are positive.

Proof. From (8) we have $\operatorname{sign}(T(x,y)) = -\operatorname{sign}(g(0)y)$, i.e. $T(x,y) > 0$, iff $\operatorname{sign}(y) = -\operatorname{sign}(g(0))$ regardless of the value of x. The latter leads to (22). From (15) we conclude that $\operatorname{sign}(\widetilde{T}(x,y)) = -\operatorname{sign}(f'_y(0)xy)$, i.e. $\widetilde{T}(x,y) > 0$, iff $\operatorname{sign}(x) = -\operatorname{sign}(f'_y(0)y)$, which gives (21).

2) $X(t,x,y) \neq -x$ when $t \in (0, T(x,y) + \widetilde{T}(x,y))$.

Proof. $X(t,x,y) \neq -x$ for all $t \in [T(x,y), \widetilde{T}(x,y))$ by the uniqueness of $\widetilde{T}(x,y)$ ensured by lemma 2. It is therefore sufficient to prove that $X(t,x,y) \neq -x$ when $t \in (0, T(x,y))$. To have the latter it is sufficient to check that the curve $\cup_{t\in(0,T(x,y))}\{(X(t,x,y), Y(t,x,y))\}$ and the line $-x \times \mathbb{R}$ are located on different sides of the line $x \times \mathbb{R}$. This property holds, if $\operatorname{sign}(x) = -\operatorname{sign}(f(x,y))$, which is a consequence of $\operatorname{sign}(x) = -\operatorname{sign}(f'_y(0)y)$, that takes place when the signs of x and y satisfy (21) and (22).

□

2.2 The normal form of the Poincare map in the neighborhood of a fold-fold singularity

Let P_x^R be the composition $\widetilde{\mathcal{P}} \circ \mathcal{P}$ obtained by applying lemmas 1 and 2 to (1) with $i = R$. Let P_x^L be the composition $\widetilde{\mathcal{P}} \circ \mathcal{P}$ obtained by applying lemmas 1 and 2 to (1) with $i = L$. Introduce

$$P(y) = P_{-x}^L(P_x^R(y))$$

Corollary 3 *Assume that both (1) with $i = L$ and (1) with $i = R$ satisfy the assumptions of lemmas 1 and 2. Then, for any $m > 0$ there exists $\delta > 0$ such that for all $m\sqrt[3]{|x|} \leq |y|$ and $|y| \leq \delta$ the map admits representation*

$$P(y) = y + (\alpha^L - \alpha^R)y^2 + (\beta^R - \beta^L)\frac{x}{y} + \Delta(x,y),$$

where

$$\Delta(0,0) = \lim_{|x|\leq m|y|^3,\ y\to 0} \frac{\Delta'_y(x,y)}{y} = \lim_{|x|\leq m|y|^3,\ y\to 0} \Delta'_x(x,y)y = 0.$$

If, in addition,

$$\operatorname{sign}(x) = -\operatorname{sign}({f^R}'_y(0)g^R(0)), \quad {f^R}'_y(0){f^L}'_y(0) > 0, \quad g^R(0)g^L(0) < 0, \tag{23}$$

then for

$$y \in \left\{ y : -\operatorname{sign}(g^R(0))y \in \left[m\sqrt[3]{|x|}, \delta \right] \right\}$$

the map $y \mapsto P(y)$ is the Poincaré map for switched system (1)-(2) induced by cross-section $\{x\} \times \mathbb{R}$.

Proof. *Step 1: Properties of $\Delta(x,y)$.* Corollary 2 implies that

$$\begin{aligned} P_x^R(y) &= -y + \alpha^R y^2 - \beta^R \frac{x}{y} + \Delta^R(x,y), \\ P_{-x}^L(y) &= -y + \alpha^L y^2 + \beta^L \frac{x}{y} + \Delta^L(x,y), \end{aligned}$$

where both $r(x,y) = \Delta^R(x,y)$ and $r(x,y) = \Delta^L(x,y)$ satisfy the limiting relations (20). Direct computation yields

$$\begin{aligned} \Delta(x,y) = {} & \Delta^L(x,y) - \Delta^R(x,y) + \alpha^L\Delta^R(x,y)^2 + 2\alpha^L\beta^R x + \alpha^L(\beta^R)^2\frac{x^2}{y^2} - 2\alpha^L\beta^R\Delta^R(x,y)\frac{x}{y} - \\ & -2\alpha^L\Delta^R(x,y)y - 2\alpha^L\alpha^R\beta^R xy + 2\alpha^L\alpha^R\Delta^R(x,y)y^2 - 2\alpha^L\alpha^R y^3 + \alpha^L(\alpha^R)^2y^4 + \\ & + \frac{\beta^L x}{\Delta^R(x,y) - \beta^R(x/y) - y + \alpha^R y^2} + \beta^L\frac{x}{y} + \Delta^L\left(x, -y + \alpha^R y^2 - \beta^R\frac{x}{y} + \Delta^R(x,y)\right) \end{aligned}$$

and the required properties of Δ follow from (20).

Step 2: The relation between map P and the flow of switched system (1)-(2). According to corollary 2 the map P_x^R is a point transformation for (1) with $i = R$ on $\left\{y : -\text{sign}(g(0))y \in \left[m\sqrt[3]{|x|}, \delta\right]\right\}$, where $\text{sign}(x) = \text{sign}\left((f^R)'_y(0)g^R(0)\right)$.

Map P is a Poincaré map for (1)-(2), if the following two properties hold.

1) P_{-x}^L needs to be a point transformation for (1) with $i = L$ from a subset of $\{-x\} \times \mathbb{R}$ to $\{x\} \times \mathbb{R}$.

   Proof. Based on corollary 2, it suffices to have $\text{sign}(-x) = -\text{sign}\left((f^L)'_y(0)g^L(0)\right)$ for 1) to hold. This sufficient property follows from (23).

2) the subset from 1) needs to be at least $P_x^R\left(\left\{y : -\text{sign}(g(0))y \in \left[m\sqrt[3]{|x|}, \delta\right]\right\}\right)$.

   Proof. Corollary 2 implies that the range of P^R is contained in the domain of P^L, if $g^R(0)g^L(0) < 0$, which is one of our assumption (23).

□

2.3 The dynamics of the map $\mathbf{P(y) = y + \alpha y^2 + \beta \frac{x}{y}}$.

Proposition 1 *Consider a map*

$$P(y) = y + \alpha y^2 + \beta \frac{x}{y} + R(x, y),$$

such that, for any $m > 0$, the C^2 in $0 < m\sqrt[3]{|x|} < |y|$ reminder R verifies

$$R(0,0) = \lim_{|x| \le m|y|^3,\ y \to 0} \frac{R'_y(x,y)}{y} = \lim_{|x| \le m|y|^3,\ y \to 0} R'_x(x,y)y = 0.$$

If $\alpha\beta \neq 0$, then for any $m > \frac{|\alpha|}{|\beta|}$ there exist $\delta > 0$ and $\gamma > 0$ such that for all $|x| < \gamma$ the map P admits a unique fixed point $y(x)$ in the set

$$\left[-\delta, -\sqrt[3]{|x|/m}\right] \cup \left[\sqrt[3]{|x|/m}, \delta\right] =: I_{-1} \cup I_1.$$

Moreover,

$$\frac{x}{y(x)^3} \to -\frac{\alpha}{\beta} \quad \text{as} \quad x \to 0, \tag{24}$$

in particular,

$$y(x) \in I_{\text{sign}(-\alpha\beta x)}. \tag{25}$$

If $x\beta < 0$, then $y(x)$ is an attractor whose domain of attraction is at least $I_{\text{sign}(-\alpha\beta x)}$. If $x\beta > 0$, then $y(x)$ is a repeller and each trajectory that originates in $I_{\text{sign}(-\alpha\beta x)} \setminus \{y(x)\}$ leaves $I_{\text{sign}(-\alpha\beta x)}$ in finite time.

Proof. *Step 1. The existence and uniqueness of $x(y)$.* The equation $P(y) = y$ is equivalent to $F(x, y) = 0$, where

$$F(x, y) = \alpha y^3 + \beta x + R(x, y)y.$$

We have $F(0, 0) = 0$ and $F'_x(x, y) \to \beta$ as $|x| \le m|y|^3$, $y \to 0$. Therefore, there exist $\delta > 0$ such that for all $|y| < \delta$ the equation $F(x, y) = 0$ has a unique solution $|x(y)| \le m|y|^3$.

Step 2. Strict monotonicity $x(y)$. Differentiating $F(x(y), y) = 0$ and expressing $x'(y)$, one concludes

$$\lim_{y \to 0} \frac{x'(y)}{y^2} = -\frac{3\alpha}{\beta}.$$

Therefore we can assume that $\delta > 0$ is so small that $x(y)$ is strictly monotone on $[-\delta, \delta]$. Defining $\gamma = \min\{|x(-\delta)|, |x(\delta)|\}$ we conclude that for any $|x| \le \gamma$ the equation $F(x, y)$ has a unique solution $y \in I_{-1} \cup I_1$ given by $y = x^{-1}(x)$. Put $y(x) = x^{-1}(x)$.

Step 3. Local stability of $y(x)$. Computing the derivative of $P'(y(x))$ we conclude that $y(x)$ is an attractor or repeller according to whether $2\alpha y(x) - \beta\dfrac{x}{y(x)^2} + R'_y(x, y(x)) < 0$ or $2\alpha y(x) - \beta\dfrac{x}{y(x)^2} + R'_y(x, y(x)) > 0$. Considering $y(x) > 0$, we divide these two inequalities by $y(x)$ to obtain

$$\text{attractor}: \qquad 2\alpha - \beta\frac{x}{y(x)^3} + \frac{R'_y(x, y(x))}{y(x)} < 0 \tag{26}$$

$$\text{repeller}: \qquad 2\alpha - \beta\frac{x}{y(x)^3} + \frac{R'_y(x, y(x))}{y(x)} > 0 \tag{27}$$

Based on (24), the inequalities (26) and (27) converge to $\alpha < 0$ and $\alpha > 0$ respectively as $x \to 0$.

For those values of x for which $y(x) < 0$, the inequalities (26) and (27) flip the signs and $y(x)$ turns out to be an attractor or a repeller according to whether $\alpha > 0$ or $\alpha < 0$.

Combining the condition on the sign of α with the condition (25) on the sign of $y(x)$ we conclude that $y(x)$ is an attractor or a repller according to whether $\beta x > 0$ or $\beta x < 0$.

Step 4. Attractivity of $y(x)$ *in* $I_{\text{sign}(-\alpha\beta x)}$. Let $\delta, m > 0$ be some constants for which the conclusions of steps 1-3 hold. Let $\delta_1, m_1 > 0$, $0 < \delta_1 < \delta$ and $m > m_1 > \dfrac{|\alpha|}{|\beta|}$, be another pair of constants for which the conclusions of steps 1-3 hold too. The proof of attractivity is split in 3 parts.

A: $P(I_{m_1,\delta_1}(x)) \subset I_{m,\delta}(x)$ for $\delta_1, \gamma > 0$ sufficiently small.

Proof. Fix some $y \in I_{m_1,\delta_1}(x)$ and first consider the case when $y \geq \sqrt[3]{|x|/m_1}$. Then

$$\begin{aligned} P(y) &= y\left(1 + \alpha y + \beta\frac{x}{y^2} + \frac{\Delta(x,y)}{y}\right) \geq \\ &\geq \sqrt[3]{|x|/m} \cdot \frac{\sqrt[3]{m}}{\sqrt[3]{m_1}}\left|1 - |\alpha|\sqrt[3]{|x|/m_1} - |\beta|\frac{|x|}{\sqrt[3]{|x|^2/m_1^2}} - \left|\frac{\Delta(x, \sqrt[3]{|x|/m_1})}{\sqrt[3]{|x|/m_1}}\right|\right|. \end{aligned}$$

Therefore, if $k \in (1, \sqrt[3]{m}/\sqrt[3]{m_1})$, then $P(y) \geq \sqrt[3]{|x|/m} \cdot k > \sqrt[3]{|x|/m}$ for $\gamma > 0$ sufficiently small.

If $y \leq \delta_1$ then we have $P(y) \leq \delta_1 \left|1 + \alpha y + \beta x/y^2 + \Delta(x,y)/y\right| < \delta$, when $\delta_1 > 0$ is small enough.

The case when $y < 0$ can be consider by analogy.

B: If $y \in I_{m,\delta(x)} \backslash I_{m_1,\delta_1}(x)$ then $y < y(x) \Longrightarrow P(y) < y(x)$ and $y > y(x) \Longrightarrow P(y) > y(x)$.

Proof. Consider $y > 0$. Let us show that $P(y) < y(x)$ for any $\sqrt[3]{|x|/m} \leq y \leq \sqrt[3]{|x|/m_1}$. We have

$$\begin{aligned} \frac{P(y)}{y(x)} &= \left|\frac{y + \alpha y^2 + \beta x/y + R(x,y)}{y(x)}\right| \leq \\ &\leq \sqrt[3]{\frac{x}{y(x)^3}} \cdot \frac{1}{\sqrt[3]{m_1}}\left(1 + |\alpha|\sqrt[3]{|x|/m_1} + |\beta|\frac{|x|}{\sqrt[3]{|x|^2/m_1^2}} + \left|\frac{\Delta(x, \sqrt[3]{|x|/m_1})}{\sqrt[3]{|x|/m_1}}\right|\right). \end{aligned}$$

Since $m_1 > |\alpha|/|\beta|$, then $\sqrt[3]{x/y(x)^3} \cdot \sqrt[3]{1/m_1} \to k < 1$ as $x \to 0$. Therefore, $P(y)/y(x) < 1$, if $\gamma > 0$ is sufficiently small. In the case where $\delta_1 \leq y \leq \delta$ the fact $P(y) > y(x)$ follows from $P(y) - y = \alpha y^2 + \beta x/y + R(x,y)$ straightaway.

The case when $y > 0$ can be considered by analogy.

C: If $y \in I_{m,\delta}(x) \backslash I_{m_1,\delta_1}(x)$ then $|P(y) - y(x)| < |y - y(x)|$.

Proof. Consider the case where $y(x) > 0$. Let $0 < y < y(x)$. From B we have that $P(y) < y(x)$, so it is enough to show that $y < P(y)$. Assume the contrary, i.e. that $P(y) < y$. Step 3 implies that there exists $y < y_* < y(x)$ such that $P(y_*) > y_*$, which implies the existence of $y_0 \in (y, y_*)$ such that $P(y_0) = y_0$. This contradicts the uniqueness of $y(x)$ established in Steps 1-2. The case where $y > y(x)$ and the case where $y(x) < 0$ can be considered by analogy.

Parts A and C imply that $P(I_{m,\delta}) \subset I_{m,\delta}$, which imply that given any $y \in I_{m,\delta}$ the sequence $P^n(y)$ converges to the unique fixed point $y(x)$.

The repelling statement can be proved by analogy. □

2.4 The main theorem

Theorem 1 *Let the C^4 functions f^L, g^L, f^R, f^L satisfy (4), i.e. the origin is a fold-fold singularity of the reduced system (3). Assume that*

$$f^{R\prime}{}_y(0)f^{L\prime}{}_y(0) > 0, \;\; g^R(0)g^L(0) < 0, \tag{28}$$

$$ag^R(0) \neq 0, \quad \text{where} \quad a = \alpha^L - \alpha^R \;\text{ with }\; \alpha^{L,R} = 2\frac{f^{L,R\prime}{}_x(0) + g^{L,R\prime}{}_y(0)}{g^{L,R}(0)} + \frac{f^{L,R\prime\prime}{}_{yy}(0)}{f^{L,R\prime}{}_y(0)}, \tag{29}$$

$$abf^{R\prime}{}_y(0) < 0, \quad \text{where} \quad b = \beta^R - \beta^L \;\text{ with }\; \beta^{L,R} = -\frac{2g^{L,R}(0)}{f^{L,R\prime}{}_y(0)}. \tag{30}$$

Then, for any $m > |a|/|b|$ there exists $\delta > 0$ such that for $xf^{R\prime}{}_y(0)g^R(0) < 0$ sufficiently close to zero, the switched system (1)-(2) admits a unique $y \in J = \left\{y : -\text{sign}\,(g^R(0))y \in \left[m\sqrt[3]{|x|}, \delta\right]\right\}$ such that (x, y) is the initial condition of a limit cycle of (1)-(2). If $(x(t), y(t))$ is any solution of (1)-(2) with $(x(0), y(0)) \in \{x\} \times J$ then the sequence $\bigcup\limits_{t\geq 0} (\{x(t), y(t)\} \cap \{x\} \times J)$ accumulates at (x, y) or contains only a finite number of elements according to whether $ag^R(0) > 0$ or $ag^R(0) < 0$. In particular, the limit cycle is orbitally stable, if $ag^R(0) > 0$, and unstable, if $ag^R(0) < 0$.

The conclusion follows by combining the statements of corollary 3 and proposition 1.

3 Applications

3.1 Switched mass-spring oscillator

A toy model that illustrates the essence of our main result is the simple mass-spring oscillator

$$\begin{aligned} \dot{x}(t) &= y(t), \\ \dot{y}(t) &= -x(t) - cy(t) + d, \end{aligned} \tag{31}$$

whose damping c and forcing d switch from (c^L, d^L) to (c^R, d^R) and back. As a consequence, the unique (globally attracting) equilibrium of this oscillator alternates between $(x, y) = (d^L, 0)$ and $(x, y) = (d^R, 0)$, see fig. 3.1. We will equip oscillator (31) with the following switching law

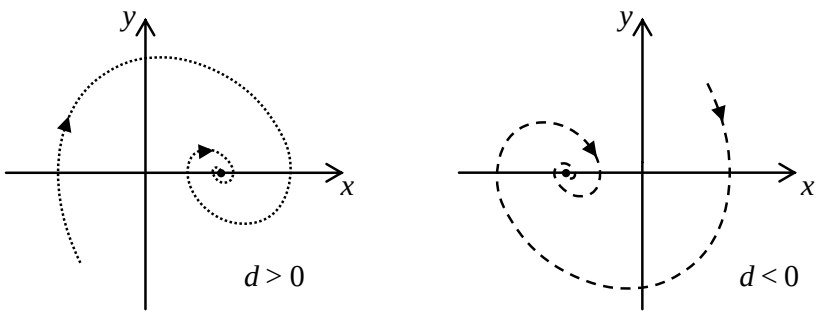

Figure 3: Sample trajectories of (31) for positive and negative values of d.

$$\begin{aligned} (c, d) &:= (c^L, d^L), \quad \text{if } x(t) = -x, \\ (c, d) &:= (c^R, d^R), \quad \text{if } x(t) = x. \end{aligned} \tag{32}$$

The switching law (32) can be, for example, executed by switching magnets (assuming that the mass in the oscillator (31) is metal) which are connected with sensors positioned at coordinates $-x$ and x, see [8].

Proposition 2 *Consider $c^L > 0$, $c^R > 0$, $d^L d^R < 0$ and introduce*

$$a = -2\frac{c^L}{d^L} + 2\frac{c^R}{d^R}, \quad b = -2d^R + 2d^L.$$

Then, for any $m > |a|/|b|$ and for $xd^R < 0$ sufficiently close to zero, the switched mass-spring oscillator (31)-(32) admits a unique limit cycle with the initial condition $(x, y(x)) \to 0$ as $x \to 0$ and $|x|/|y(x)|^3 \leq m$. The limit cycle is orbitally stable.

Proof. We have

$$\begin{pmatrix} f^L(0) \\ g^L(0) \end{pmatrix} = \begin{pmatrix} 0 \\ d^L \end{pmatrix}, \begin{pmatrix} f^{L\prime}(0) \\ g^{L\prime}(0) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -c^L \end{pmatrix}, \begin{pmatrix} f^R(0) \\ g^R(0) \end{pmatrix} = \begin{pmatrix} 0 \\ d^R \end{pmatrix}, \begin{pmatrix} f^{R\prime}(0) \\ g^{R\prime}(0) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -c^R \end{pmatrix}.$$

The two inequalities in (28) become $1 > 0$ and $d^L d^R < 0$. The constants and the inequality in (29) turn out to be

$$\alpha^L = 2\frac{-c^L}{d^L}, \quad \alpha^R = 2\frac{-c^R}{d^R}, \quad ad^R \neq 0.$$

The constants and the inequality in (30) evaluate to

$$\beta^L = -2d^L, \quad \beta^R = -2d^R, \quad \left(-\frac{c^L}{d^L} + \frac{c^R}{d^R}\right)\left(-d^R + d^L\right) < 0.$$

The condition for stability leads to $ad^R > 0$ or, equivalently, to $(-d^R + d^L)d^R < 0$, which always holds. □

Proposition 2 supports the simulations carried out for (32) in [8].

3.2 Anti-lock braking system

The analysis in this section concerns the *single-corner* model (also known as *quater-car* model). This model is typically used for the preliminary design of wheel braking control algorithms (see [11, 14]). If the longitudinal dynamics of the vehicle is much slower than the rotational dynamics of the wheel than the interplay between the longitudinal slip and the braking torque of the wheel reads as (see [14])

$$\dot{\lambda}(t) \quad = \quad -F(\lambda(t)) + F_0 T_b(t), \tag{33}$$

with

$$F(\lambda) = \frac{1}{\nu}\left(\frac{1-\lambda}{m} + \frac{r^2}{J}\right) F_z \mu(\lambda), \quad \mu(\lambda) = \theta_{r1}(1 - \exp(-\lambda\theta_{r2})) - \lambda\theta_{r3}, \quad F_0 = \frac{r}{\nu J},$$

where ν is the longitudinal speed of the vehicle, r is the wheel radius, J is the moment of inertia of the wheel, m is the mass of the quarter-car, F_z is the vertical force at the tire-road contact point, θ_{r1}, θ_{r2}, θ_{r3} are positive constants that reflect the road conditions.

The assumption of either a fixed or a user-selectable actuator rate limit has been used in several works on rule-based ABS, see [11, 14] and references therein. This leads to the following differential equation for T_b

$$\dot{T}_b = u, \tag{33a}$$

where u is a control input, that can take a finite number of values. In this paper we consider the simplest control logic where the actuator rates can take two values $k_1 = k$ and $k_2 = -k$ of equal modulus, see fig. 3.2 for the phase portrait of system (33)-(33a). Following [11, 14], the goal of the controller is to make $\lambda \mapsto \lambda(t)$ oscillating periodically within the interval $[\lambda_0 - \Delta\lambda, \lambda_0 + \Delta\lambda]$, where λ_0 is the measured wheel slip and $\Delta\lambda$ is the measurement error, i.e. the deviation of the measured wheel slip from the optimal value (that ensures the fastest braking). Whenever $\lambda(t)$ hits $\lambda_0 + \Delta\lambda$, the change of the braking torque $\dot{T}_b = u$ shall switch from $u = k_1 > 0$ to $u = k_2 < 0$. On the contrary, as soon as $\lambda_0 - \Delta\lambda$ is hit, it shall switch from $u = k_2$ back to $u = k_1$. This control logic will be shortly formulated as

$$\dot{T}_b(t) \quad := \quad \begin{cases} -k, & \text{if } \lambda(t) \geq \lambda_0 + \Delta\lambda, \\ k, & \text{if } \lambda(t) \leq \lambda_0 - \Delta\lambda. \end{cases} \tag{34}$$

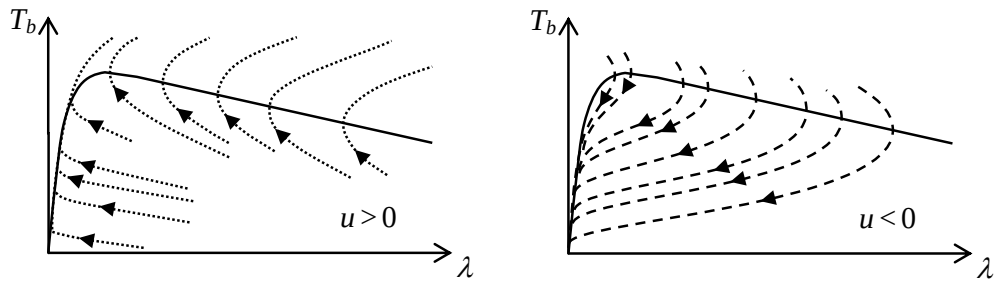

Figure 4: Trajectories of system (33)-(33a) for $k < 0$ and $k > 0$. The solid line is the graph of the function $T_b = \frac{1}{F_0}F(\lambda)$, i.e. the set of (λ, T_b) where the first component of the right-hand-side of (33)-(33a) vanishes.

Proposition 3 *Assume that $F_0 k \neq 0$. Fix $\lambda_0 \in \mathbb{R}$ and consider*

$$a = 4\frac{F'(\lambda_0)}{k}, \quad b = -4\frac{k}{F_0}.$$

If $F'(\lambda_0) > 0$ then for any $m > |a|/|b|$ and for $\Delta\lambda \cdot k > 0$ sufficiently close to zero, the anti-lock braking system (33)-(34) admits a unique limit cycle with the initial condition $(x, y(x)) \to 0$ as $x \to 0$ and $|x|/|y(x)|^3 \leq m$. The limit cycle is orbitally stable.

Proof. The first component of the right-hand-side vanishes when $T_b = (1/F_0)F(\lambda)$. Therefore, points of the type $(\lambda, F_0^{-1}F(\lambda))$ are potential candidates to produce limit cycles of (33). We have

$$\begin{pmatrix} f^L(\lambda, F_0^{-1}F(\lambda)) \\ g^L(\lambda, F_0^{-1}F(\lambda)) \end{pmatrix} = \begin{pmatrix} 0 \\ -k \end{pmatrix}, \quad \begin{pmatrix} f^{L\prime}(\lambda, F_0^{-1}F(\lambda)) \\ g^{L\prime}(\lambda, F_0^{-1}F(\lambda)) \end{pmatrix} = \begin{pmatrix} -F'(\lambda) & F_0 \\ 0 & 0 \end{pmatrix},$$
$$\begin{pmatrix} f^R(\lambda, F_0^{-1}F(\lambda)) \\ g^R(\lambda, F_0^{-1}F(\lambda)) \end{pmatrix} = \begin{pmatrix} 0 \\ k \end{pmatrix}, \quad \begin{pmatrix} f^{R\prime}(\lambda, F_0^{-1}F(\lambda)) \\ g^{R\prime}(\lambda, F_0^{-1}F(\lambda)) \end{pmatrix} = \begin{pmatrix} -F'(\lambda) & F_0 \\ 0 & 0 \end{pmatrix}.$$

The two inequalities in (28) become $(F_0)^2 > 0$ and $-k^2 < 0$. The constants and the inequality in (29) turn out to be

$$\alpha^L = 2\frac{-F'(\lambda)}{k}, \quad \alpha^R = 2\frac{-F'(\lambda)}{-k}, \quad a = -4\frac{F'(\lambda)}{k}, \quad F'(\lambda) \neq 0.$$

The constants and the inequality in (30) evaluate to

$$\beta^L = -\frac{2k}{F_0}, \quad \beta^R = -\frac{-2k}{F_0}, \quad b = \frac{4k}{F_0}, \quad -F'(\lambda) < 0,$$

which implies that the stability condition holds true. □

4 Appendix

4.1 Partial derivatives of the general solution at a fold singularity

Let $t \mapsto (X(t,x,y), Y(t,x,y))$ be the general solution of (5) that satisfies $f(0) = 0$. Then

$$\begin{aligned}
&X'_t(0) = 0,\\
&X'_d(0) = 1,\\
&X'_u(0) = 0,\\
&Y'_t(0) = g(0),\\
&Y'_d(0) = 0,\\
&Y'_u(0) = 1,\\
&Y''_{tx}(0) = f'_x(0),\\
&X''_{ty}(0) = f'_y(0),\\
&Y''_{tx}(0) = g'_x(0),\\
&Y''_{ty}(0) = g'_y(0),\\
&X''_{tt}(0) = f'_y(0)g(0),\\
&Y''_{tt}(0) = g'_y(0)g(0),\\
&X'''_{tyy}(0) = f'''_{yy}(0)\\
&X'''_{tty}(0) = f'_x(0)f'_y(0) + f'''_{yy}(0)g(0) + f'_y(0)g'_y(0)\\
&X'''_{ttt}(0) = \left[f'_x(0)f'_y(0) + f'''_{yy}(0)g(0) + f'_y(0)g'_y(0)\right] g(0)
\end{aligned} \tag{35}$$

4.2 Implicit function theorem for functions with vanishing domain

Theorem 2 *Consider $F \in C^1(\mathbb{R}\times\mathbb{R}\times\mathbb{R},\mathbb{R})$ such that $F(0,0,0) = 0$. Assume that there exist $M > 0$, $\delta_0 > 0$ and $\gamma(u) \to 0$ as $u \to 0$, such that*

$$|F(0,d,u)| \le M|u|, \qquad \text{for any } \ |d| \le \gamma(u)|u^2|,\ |u| \le \delta_0. \tag{36}$$

If

$$F'_t(t,d,u) \to L \neq 0, \quad \text{as } \ |t| \le \frac{M}{Lq}|u|,\ |d| \le \gamma(u)|u^2|,\ u \to 0, \tag{37}$$

then, for some $\delta \in (0,\delta_0)$, the equation

$$F(t,d,u) = 0 \tag{38}$$

has a unique solution $|t(d,u)| \le \dfrac{M}{Lq}|u|$ in the set

$$|d| \le \gamma(u)|u^2|, \ \ |u| \le \delta. \tag{39}$$

Moreover, the function d is differentiable in the interior of this region.

Proof. 1) Existence. Consider

$$A_{(d,u)}(t) = t - \frac{1}{L}F(t,d,u).$$

We want to show that for any $\delta > 0$ sufficiently small the map $A_{(d,u)}$ maps the disk $|t| \le \dfrac{M}{Lq}\delta$ into itself. We have

$$A_{(d,u)}{}'(t) = \frac{1}{L}\left(L - F'_t(t,d,u)\right).$$

Diminishing $\delta > 0$ so that

$$\left|A_{(d,u)}{}'(t)\right| \leq 1-q, \quad \text{for any } |t| \leq \frac{M}{Lq}|u|,\ |d| \leq \gamma(u)|u^2|,\ |u| \leq \delta,$$

one gets

$$\left|A_{(d,u)}(t)\right| \leq \left|A_{(d,u)}(0)\right| + \left|A_{(d,u)}(t) - A_{(d,u)}(0)\right| \leq \frac{1}{L}M\delta + (1-q)\frac{M}{Lq}\delta = \frac{M}{Lq}\delta.$$

2) Differentiability. Let (d_0, u_0) be from the interior of (39). Equation (38) can be solved in t near (d_0, u_0) because the conditions of the standard implicit function theorem (see e.g. [16, § 8.5.4, Theorem 1]) hold at $(t(d_0, u_0), d_0, u_0)$. Differentiability of t at (d_0, u_0) is, therefore, follows from the classical result on the differentiability of the implicit function (see same source [16, § 8.5.4, Theorem 1]).

□

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