Complex Structures in Algebra, Geometry, Topology, Analysis and Dynamical Systems

Zalman Balanov (University of Texas at Dallas) [1]

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[1] joint work with Y. Krasnov (Bar Ilan University)

1. OUTLINE

• **Five problems:**

(i) Existence of bounded solutions to quadratic ODEs;

(ii) "Fundamental Theorem of Algebra" in non-associative algebras;

(iii) "Intermediate Value Theorem" in $\mathbb{R}^2$;

(iv) Existence of maps with positive Jacobian;

(v) Surjectivity of polynomial maps.

• **Quadratic ODEs of natural phenomena:**

(i) Euler equations (solid mechanics);

(ii) Kasner equations (gen. relativity theory);

(iii) Volterra equation (population dynamics);

(iv) Aris equations (second order chemical reactions);

(v) Ginzburg-Landau nonlinearity (superconductivity);

(vi) Geodesic equation.

• **Complex structures in algebras as a common root of the above 5 problems**

• **Applications**

2.1. Bounded solutions to quadratic systems.

*"Undergraduate case".*Assume $A : \mathbb{R}^n \to \mathbb{R}^n$ is a linear operator and consider the system

$$\frac{dx}{dt} = Ax. \tag{1}$$

Proposition 1

System (1) has a periodic solution iff the following non-hyperbolicity condition is satisfied:

Condition (A): A has a purely imaginary eigenvalue.

Assume now that we are given a quadratic system

$$\frac{dx}{dt} = Q(x) \tag{2}$$

with $Q : \mathbb{R}^n \to \mathbb{R}^n$ a homogeneous (polynomial) map of degree 2 (i.e $Q(\lambda x) = \lambda^2 Q(x)$ for all $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}^n$ or, that is the same, the coordinate functions of Q are quadratic forms in n variables).

QUESTION A. What is an analogue of Condition(A) (non-hyperbolicity) for the quadratic system (2) in the context relevant to Proposition 1 (existence of bounded/periodic solutions)?

2.2. Fundamental theorem of algebra in non-associative algebras

Undergraduate fact: any complex polynomial

$$f(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n \quad (n > 0)$$

has at least one root $z_o \in \mathbb{C}$, i.e. $f(z_o) = 0$.

bf Remark. From the algebraic viewpoint, the set $\mathbb{C}$ of complex numbers has the following properties:
(i) it is a real 2-dimensional vector space;
(ii) elements of $\mathbb{C}$ can be multiplied in such a way that

$$a(\alpha b + \beta c) = \alpha ab + \beta ac \quad \forall a, b, c \in \mathbb{C}; \alpha, \beta \in \mathbb{R} \tag{3}$$

and

$$ab = ba \quad \forall a, b \in \mathbb{C}. \tag{4}$$

Definition 2

Any n-dimensional real vector space equipped with the commutative bi-linear multiplication (see (3) and (4)) is called a (commutative) algebra.

Remarks. (i) The above definition does NOT require from an algebra to be associative.
(ii) By obvious reasons, given a commutative real two-dimensional algebra A, one cannot expect that any polynomial equation in A has a (non-zero) root.

QUESTION B: Let A be a commutative real two-dimensional algebra. To which extent should be A close to $\mathbb{C}$ to ensure that a "reasonable" polynomial equation in A has a (non-zero) root?

2.3. "Intermediate Value Theorem in $\mathbb{R}^2$

Undergraduate fact (Intermediate Value Theorem): Assume:
(i) $f : [a, b] \to \mathbb{R}$ is a continuous function;
(ii) $f(a) \cdot f(b) < 0$.
Then, the equation

$$f(x) = 0$$

has at least one solution.

Question: Given a continuous map $\Phi : B \to R^2$, where B stands for a closed disc in $\mathbb{R}^2$, what is an analogue of condition (ii) providing that the equation

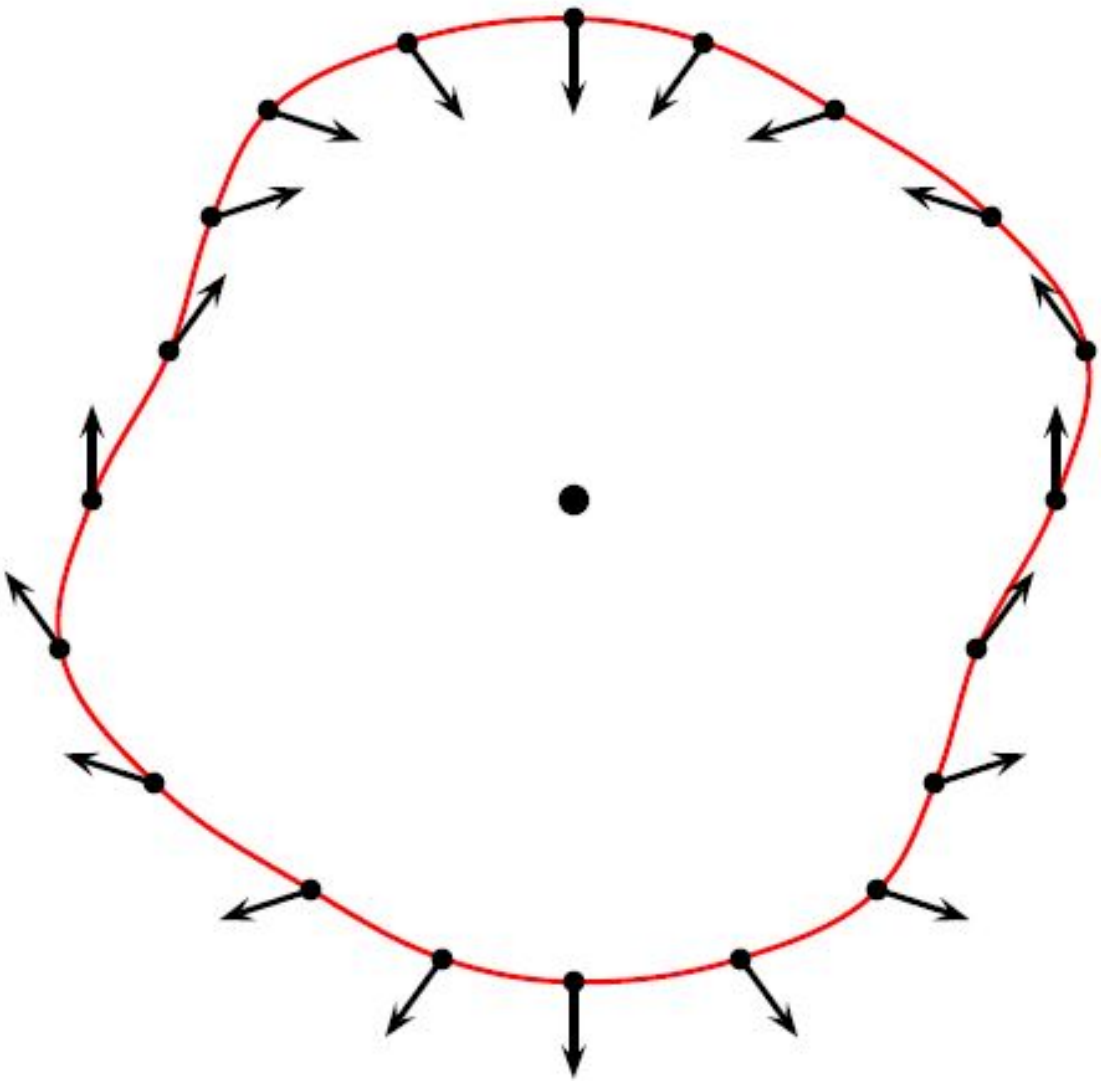
$$\Phi(u) = 0$$

has at least one solution?

Remarks.

(i) The above map Φ assigns to each $u \in B$ a vector $\Phi(u)$.

(ii) Denote by Γ the boundary of B and assume $\Phi(u) \neq 0$ for all $u \in \Gamma$. Choose a point $M \in \Gamma$ and force it to travel along Γ and to return back. Since: (a) Γ is a closed curve, and (b) Φ is a continuous vector field, the vector $\Phi(M)$ will make an integer number of rotations (called topological index and denoted by $\gamma(\Phi, \Gamma)$).



Proposition 3

Assume:
(i) $\Phi : B \to \mathbb{R}^2$ *is a continuous map with no zeros on* Γ*;*
(ii) $\gamma(\Phi, \Gamma) \neq 0$*. Then, the equation*

$$\Phi(u) = 0$$

has at least one solution inside B*.*

Question C. *Which requirements on* Φ *do provide condition (ii) from Proposition 3?*

2.4. Existence of maps with positive Jacobian determinant

Undergraduate fact: Let $f : \mathbb{C} \to \mathbb{C}$ defined by

$$f(x + iy) = u(x, y) + iv(x, y)$$

be a complex analytic map (i.e. the Cauchy-Riemann conditions

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

are satisfied). Then, the Jacobian determinant $Jf(x, y)$ is non-negative for all $x + iy \in \mathbb{C}$.

Definition 4

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a (real) smooth map. We call f positively quasi-conformal (resp. negatively quasi-conformal) if $Jf(x, y) > 0$ (resp. $Jf(x, y) < 0$) for all $(x, y) \in \mathbb{R}^2$.

Question D. Do there exist easy to verify conditions on f providing its positive/negative quasi-conformness?

2.5. Surjective quadratic maps

Obvious observation: 1-dimensional case.

Let $f : \mathbb{R} \to \mathbb{R}$ be a quadratic map, i.e. $f(x) = ax^2$, $a \in \mathbb{R}$. Then, f is **not** surjective.

Question E.

Let $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ be a quadratic map, i.e. its coordinate functions are quadratic forms in n variables. Under which conditions is Φ surjective?

3.6. Summing up:

We arrive at the following

Main Question: What is the connection between Question A (Quadratic differential systems), Question B (algebra), Question C (topology), Question D (geometric analysis), and Question E (algebraic(?) geometry or "geometric" algebra)?

Main goal of my talk: To answer the Main Question.

By-product: to illustrate the obtained results with applications to quadratic systems of practical meaning.

3. EXAMPLES OF QUADRATIC ODEs of REAL LIFE PHENOMENA

3.1. Euler equations (see [Arnold])

$$\begin{cases} \dot{\omega}_1 = ((I_3 - I_2)/I_1)\omega_2\omega_3 \\ \dot{\omega}_2 = ((I_1 - I_3)/I_2)\omega_1\omega_3 \\ \dot{\omega}_3 = ((I_2 - I_1)/I_3)\omega_1\omega_2 \end{cases}$$

describes the motion of a rotating rigid body with no external forces (here the (non-zero) principal moments of inertia I_j satisfy $I_1 \neq I_2 \neq I_3 \neq I_1$ and I_j stands for the j-th component of the angular velocity along the principal axes).

3.2. Kasner equations (see [Kasner,KinyonWalcher])

$$\begin{cases} \dot{x} = yz - x^2 \\ \dot{y} = xz - y^2 \\ \dot{z} = xy - z^2 \end{cases}$$

describe the so-called Kasner's metrics being the exact solution to the Einstein's general relativity theory equations in vacuum under special assumptions.

3.3. Volterra equations (see [HofbauerSigmund])

$$\begin{cases} \dot{x}_1 = x_1 L_1(x_1, ..., x_n) \\ \dot{x}_2 = x_2 L_2(x_1, ..., x_n) \\ ... \\ \dot{x}_n = x_n L_n(x_1, ..., x_n) \end{cases}$$

where $L_i(x_1, ..., x_n)$ is a linear non-degenerate form (also known as predator-pray equations), describe dynamics of biological systems in which n species interact.

3.4. Aris equations (see [Aris])

$$\begin{cases} \dot{x} = a_1 x^2 + 2a_2 xy + a_3 y^2 \\ \dot{y} = b_1 x^2 + 2b_2 xy + b_3 y^2 \end{cases}$$

describe a dynamics of the so-called second order chemical reactions (i.e. the reactions with a rate proportional to the concentration of the square of a single reactant or the product of the concentrations of two reactants).

3.5. More "academic" examples

3.5.1 Given $\lambda, \mu, c \in \mathbb{R}$ with $c \neq 0$, define a system (see [KinyonSagle]) "typical among quadratic three-dimensional ones admitting (non-zero) periodic solutions compatible with derivations of the corresponding fields.

$$\begin{cases} x_0 = \lambda x_0^2 + (x_1^2 + x_2^2) \\ x_1 = -2cx_0x_2 \\ x_2 = 2cx_0x_1 \end{cases}$$

3.5.2 Let H be the 4-dimensional algebra of quaternions. Given $q = q_0 + q_1 i + q_2 j + q_3 k \in \mathbb{H}$, define a conjugate to q by $\overline{q} := q_0 - q_1 i - q_2 j - q_3 k$. The following ODEs were considered in [MawhinCampos]

$$\dot{q} = \|q\|^\alpha q^\beta \overline{q}^\gamma, \quad q \in \mathbb{H}, \quad 1 \leq \alpha + \beta + \gamma.$$

If $\alpha = 2$, $\beta = 1$ and $\gamma = 0$, then one obtains a type of nonlinearities arising in Ginzburg-Landau equation which comes from the theory of superconductivity (cf. [B'ethuelBrezisH'elein]).

3.6. Geodesic equations.

Let Φ be a regular surface in $\mathbb{R}^3$ parametrized by

$$r(u,v) = (x(u,v), y(u,v), z(u,v)).$$

Put

$$E = E(u,v) := x_u^2 + y_u^2 + z_u^2,$$
$$F = F(u,v) := x_u x_v + y_u y_v + z_u z_v,$$
$$G = G(u,v) := x_v^2 + y_v^2 + z_v^2$$

and take the so-called First Fundamental Form

$$dr^2 := Edu^2 + 2Fdudv + Gdv^2.$$

Properties of Φ depending only on dr^2 constitute the so-called intrinsic geometry of Φ. In particular, an important problem of intrinsic geometry is to study the behavior of geodesic curves on Φ, i.e. solutions to the differential system:

$$\begin{cases} u'' = -\frac{E_u}{2E} u'^2 - \frac{E_v}{E} u'v' + \frac{G_u}{2E} v'^2 \\ v'' = \frac{E_v}{2E} u'^2 - \frac{G_u}{G} u'v' + \frac{G_v}{2G} v'^2 \end{cases} \tag{5}$$

Remarks. (i) In general, one cannot integrate the second order differential system (5).
(ii) System (5) is quadratic with respect to the velocity (u', v').
(iii) In contrast to the above examples, where quadratic systems were considered in a fixed linear space, one should consider (5) as a family of systems depending on (u, v).

3.7. Asymptotically homogeneous systems

Finally, together with all the above examples, one can associate external perturbations of practical meaning leading to the systems of the form

$$\frac{dx}{dt} = Q(x) + h(t, x), \tag{6}$$

where $x \in \mathbb{R}^n$, $Q : \mathbb{R}^n \to \mathbb{R}^n$ is quadratic (or, more generally, homogeneous of order $k > 1$) and $h : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous, T-periodic in t and "small" in a certain sense. The problem of the existence of T-periodic solutions to system (6) will be also discussed in my talk (this problem was intensively studied by V. Nemytskiy, M. Krasnoselskiy, N. Bobylyov, E. Muhamadiyev, J. Mawhin, V. Pliss, Gomory,...

4. COMPLEX STRUCTURES IN ALGERAS AS A COMMON ROOT OF THE ABOVE ISSUES

4.1. From quadratic maps to multiplications in algebras: Riccati equation

Standard fact: Let $b : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a symmetric bilinear form. Then, the restriction to the diagonal given by

$$q(x) := b(x, x) \tag{7}$$

is a quadratic form. Conversely, let $q : \mathbb{R}^n \to \mathbb{R}$ be a quadratic form. Then, the formula

$$b(x, y) := \frac{1}{2}(q(x + y) - q(x) - q(y)) \quad (x, y \in \mathbb{R}^n) \tag{8}$$

assigns to the quadratic form q the symmetric bilinear form $b : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ in such a way that $q(x) = b(x, x)$.

Example

Take the symmetric bilinear form $b(x, y) = x_1 y_1 + x_2 y_2$, where $x = (x_1, x_2)$, $y = (y_1, y_2)$. Then, formula (7) gives the quadratic form

$$q(x) = b(x, x) = x_1^2 + x_2^2. \tag{9}$$

Conversely, take the quadratic form (9) and apply formula (8) to get the symmetric bilinear form

$$b(x, y) = \frac{1}{2}(x_1 + y_1)^2 + (x_2 + y_2)^2 (x_1^2 + x_2^2)(y_1^2 + y_2^2) = x_1 y_1 + x_2 y_2, \tag{10}$$

i.e. we pass from the square of the norm (which is q) to the inner product (which is $b(x, y)$) and vice versa.

Main passage.

Assume now that

$$B : \mathbb{R}^n \times R^n \to \mathbb{R}^n$$

is a commutative bilinear multiplication (i.e. each coordinate function of this map is a symmetric bilinear form). Then, the formula

$$Q(x) := B(x, x)$$

defines the quadratic map

$$Q : \mathbb{R}^n \to \mathbb{R}^n$$

(i.e. its coordinate functions are quadratic forms in n variables). Conversely, given a quadratic map

$$Q : \mathbb{R}^n \to \mathbb{R}^n,$$

one can use the formula

$$B(x, y) := \frac{1}{2}(Q(x + y) - Q(x) - Q(y)) \tag{11}$$

to define the commutative multiplication in $\mathbb{R}^n$.

Definitions and Notations.

(i) Given a quadratic map $Q : \mathbb{R}^n \to \mathbb{R}^n$, denote by A_Q the (real) commutative algebra with the multiplication (11) and call A_Q the algebra associated to Q.

(ii) To simplify the notations, we use the symbol $x \circ y$ instead $B(x, y)$ and x^2 instead $x \circ x$.

(iii) Any quadratic system $\frac{dx}{dt} = Q(x)$ can be rewriten in A_Q in the form

$$\frac{dx}{dt} = x^2 \quad (x \in A_Q), \tag{12}$$

and called by the obvious reason **Riccati equation** in A_Q.

(iv) By replacing everywhere "$\mathbb{R}$" with "$\mathbb{C}$", one can speak about complex algebras and complex Riccati equation.

The statement following below shows that passing from a quadratic system to the Riccati equation in the corresponding algebra is not just a formal trick!

Proposition 5

Two quadratic systems are linearly equivalent iff the algebras associated to them are isomorphic.

The Main Paradigm: The above proposition suggests to study dynamics of quadratic systems via the properties of underlying algebras - this natural idea was suggested by L. Markus in 1960.

Obvious observations

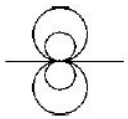
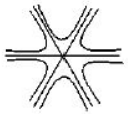
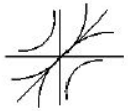
Given a quadratic map $Q : \mathbb{R}^n \to \mathbb{R}^n$ and the Riccati equation (12), one has:

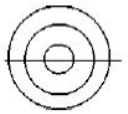
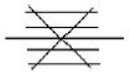
(i) equilibria to (12) coincide with 2-nilpotents in A_Q (i.e. solutions to the equation $x^2 = 0$ in A_Q);

(ii) ray solutions to (12) coincide with straight lines through idempotents in A_Q (i.e. non-zero solutions to the equation $x^2 = x$ in A_Q).

QUESTION. Given an algebra A_Q, do there exist polynomial equations in A_Q whose solubility is responsible for the existence of bounded/periodic solutions to the Riccati equation?

To get a feeling on the results we are looking for, consider phase portraits of five typical two-dimensional quadratic systems.

$\mathbb{C}$	$\dot{x} = x^2 - y^2$ $\dot{y} = 2xy$	
$\overline{\mathbb{C}}$	$\dot{x} = x^2 - y^2$ $\dot{y} = -2xy$	
$\mathbb{R} \oplus \mathbb{R}$	$\dot{x} = x^2$ $\dot{y} = y^2$	

$\mathbb{C}_\infty$	$\dot{x} = -\sqrt{2}y^2$ $\dot{y} = \sqrt{2}xy$	
$\mathbb{C}_0$	$\dot{x} = x^2 - y^2$ $\dot{y} = 0$	

4.2. Complex structures in algebras

Clearly, three of the above systems ($\mathbb{C}$, $\mathbb{C}_\infty$ and $\mathbb{C}_0$) admit bounded solutions, while two other do not admit.

Question: Which polynomial equations are (non-trivially) soluble in $\mathbb{C}$, $\mathbb{C}_\infty$ and C_0 and are **not** soluble in $\overline{\mathbb{C}}$ and $\mathbb{R} \oplus \mathbb{R}$?

Starting point: The existence of the imaginary unit $i \in \mathbb{C}$ means that the equation

$$x^2 = -1 \tag{13}$$

is soluble in $\mathbb{C}$. By the trivial reason, one CANNOT expect that equation (13) is soluble in a commutative two-dimensional algebra A even "close" to $\mathbb{C}$: usually, A does **not** contain a neutral element e (i.e. $e * a = a$ for all a). This justifies the following

Definition 6

Let A be an algebra (in general, not necessarilly associative or commutative).

(i) We say that A admits a **complex structure** if some polynomial equation generalizing (13) is non-trivially soluble in A.

(ii) A non-zero element $y \in A$ satisfying

$$y^2 * y^2 = -y^2 \tag{14}$$

is called a **negative square idempotent**.

(iii) Let A be an algebra (commutative or associative). A non-zero element $x \in A$ satisfying

$$x^3 = -x \tag{15}$$

is said to be a **negative** 3-**idempotent**.

Degenerate versions of the equations determining negative square idempotents and negative 3-idempotents:

(iv) A non-zero element $y \in A$ with $y^2 \neq 0$ is called a **square nilpotent** if (cf. (14))

$$y^2 * y^2 = 0. \tag{16}$$

(v) A non-zero element $x \in A$ is called a **3-nilpotent** if (cf. (15)

$$x^3 = 0. \tag{17}$$

Definition 7

(i) By a **complete complex structure** in an algebra B we mean the existence of a two-dimensional subalgebra in B containing both negative square idempotent and negative 3-idempotent.

(ii) By a **generalized complete complex structure** in an algebra B we mean the existence of a two-dimensional subalgebra that either admits a complete complex structure, or contains both negative 3-idempotent and square nilpotent, or contains both negative square idempotent and 3-nilpotent.

To illustrate these notions, describe explicitly the multiplications in the algebras $\mathbb{C}$, $\overline{\mathbb{C}}$, $\mathbb{C}_\infty$, $\mathbb{C}_0$ and $\mathbb{R} \oplus \mathbb{R}$ underlying 5 systems considered above.

(i) $\mathbb{C}$:

$$z_1 * z_2 := z_1 \cdot z_2;$$

(ii) $\overline{\mathbb{C}}$:

$$z_1 * z_2 := z_1 \cdot z_2;$$

(iii) $\mathbb{C}_\infty$:

$$z_1 * z_2 := \frac{i}{\sqrt{2}} \cdot (\mathrm{Im}(z_2) \cdot z_1 + \mathrm{Im}(z_1) \cdot z_2);$$

(iv) $\mathbb{C}_0$:

$$z_1 \cdot z_2 := \mathrm{Re}(z_1 \cdot z_2);$$

(where "$\cdot$" stands for the usual multiplication in $\mathbb{C}$);

(v) $\mathbb{R} \oplus \mathbb{R}$:

$$(x_1, x_2) * (y_1, y_2) := (x_1 y_1, x_2 y_2).$$

By direct computation (put in (a)–(c) $x = y = i$):

(a) $\mathbb{C}$ is the only algebra containing both negative square idempotent ($y^2 * y^2 = -y^2$) and negative 3-idempotent ($x^3 = -x$), i.e $\mathbb{C}$ admits a complete complex structure.

(b) $\mathbb{C}_\infty$ contains a negative 3-idempotent ($x^3 = -x$) together with a square nilpotent ($y^2 * y^2 = 0$), i.e. $\mathbb{C}_\infty$ admits a generalized complete complex structure.

(c) $\mathbb{C}_0$ contains a negative square idempotent ($y^2 * y^2 = -y^2$) together with a 3-nilpotent ($x^3 = 0$), i.e. $\mathbb{C}_0$ admits a generalized complete complex structure.

(d) $\overline{\mathbb{C}}$ contains a negative square idempotent ($y^2 * y^2 = -y^2$) and does **NOT** contain any other (including degenerate) complex structure.

(e) $\mathbb{R} \oplus \mathbb{R}$ does **NOT** contain any complex structure.

Main observation: In the above examples, (non-trivial) bounded solutions occur only for the Riccati equations in algebras containing generalized complex structures.

Theorem 8

Let $A = (\mathbb{R}^2, *)$ be a real commutative two-dimensional algebra. Then, the Riccati equation

$$\frac{dx}{dt} = x * x = x^2 \quad (x \in A) \tag{18}$$

admits a (non-trivial) bounded solution if and only if A admits a generalized complex structure.

Definition 9

Non-trivial bounded solutions of the type occuring in the algebra $\mathbb{C}$ (i.e. the ones starting and ending at the same point) are called homoclinic.

Main Link Theorem

Let $A = (\mathbb{R}^2, *)$ be a real commutative two-dimensional algebra without 2-nilpotents, $f : A \to A$ the quadratic map defined by $f(x) := x \cdot x = x^2$. Then, the following conditions are equivalent:
(i) the Riccati equation

$$\frac{dx}{dt} = x^2 \quad (x \in A) \tag{19}$$

admits a (bounded) homoclinic solution;
(ii) any polynomial equation

$$x^3 + p \cdot x^2 + q \cdot x = 0 \quad (x \in A) \tag{20}$$

is non-trivially soluble for all $p, q \in \mathbb{R}$ with $p^2 + q^2 \neq 0$;
(iii) $\gamma(f, \Gamma) = 2$;
(iv) f is positively quasi-conformal;
(v) A admits a complete complex structure.

5. Applications

5.1. Direct consequences.

Corollary 10

Theorem A gives a necessary and sufficient condition for the Aris model of the second order chemical reaction to have a bounded solution.

Consider the complex Volterra equations

$$\begin{cases} \dot{x}_1 = x_1 \cdot L_1(x_1, ..., x_n) \\ \dot{x}_2 = x_2 \cdot L_2(x_1, ..., x_n) \\ ... \\ \dot{x}_n = x_n \cdot L_n(x_1, ..., x_n) \end{cases}, \tag{21}$$

i.e. $x_i \in \mathbb{C}$ and $L_i(x_1, ..., x_n)$ is a $\mathbb{C}$-linear nondegenerate form.

Corollary 11

Any complex Volterra equation admits a (bounded) homoclinic solution.

Proof.

Let $Q : \mathbb{C}^n \to \mathbb{C}^n$ be the quadratic map related to the right-hand side of (21), and A_Q – the corresponding commutative algebra. By direct computation, A_Q contains an idempotent e ($e^2 = e$). Take a subalgebra $\mathbb{C}[e]$ generated by e. Since e is the idempotent, $\mathbb{C}[e]$ is isomorphic to $\mathbb{C}$.

By the Main Link Theorem, there is a homoclinic solution to (21) restricted to $\mathbb{C}[e]$. Clearly, it is a solution to the initial system as well. $\square$

5.2. Riccati equation in rank three algebras and Clifford algebras.

Definition 12

(i) Let A be an associative algebra with unit 1. A is called a Clifford algebra if there are a linear form γ_1 and a quadratic form γ_2 on A such that

$$x^2 = \gamma_1(x)x + \gamma_2(x)1 \quad \forall x \in A. \tag{22}$$

(ii) A commutative algebra A is called a rank three algebra if there are a linear form γ_1 and a quadratic form γ_2 on A such that

$$x^3 = \gamma_1(x)x^2\gamma_2(x)x \quad \forall x \in A. \tag{23}$$

Remarks. (i) A Clifford algebra is not supposed to be commutative while a rank three algebra is not supposed to be associative/containing a unit.

(ii) For both Clifford and rank three algebras, any one-generated subalgebra is of dimension ≤ 2 or, that is the same, any trajectory to the Riccati equation in Clifford or rank three algebra is **PLANAR.**

Combining Theorem A with Remark (ii) yields

Corollary 13

Let A be a rank three algebra. Then, the Riccati equation $\frac{dx}{dt} = x^2$ in A has a bounded solution iff A admits a generalized complex structure.

Corollary 14

Let A be a Clifford algebra. Then, the Riccati equation $\frac{dx}{dt} = x^2$ in A has a (non-zero) bounded solution iff A contains a subalgebra isomorphic to $\mathbb{C}$.

5.3. Periodic solutions to asymptotically homogeneous systems

Given a rank three algebra A, consider a differential system

$$\frac{dx}{dt} = x^3 + h(t,x) \quad (t \in \mathbb{R}, x \in A), \tag{24}$$

where $h : \mathbb{R} \times A \to A$ is continuous T-periodic in t. Below is a "Muhamadiyev-type result".

Corollary 15

Assume A is free from 2-nilpotents, 3-nilpotents and negative 3-idempotents. Then, for any h satisfying

$$\lim_{\|x\|\to\infty} \sup_t \|x\|^{-3}\|h(t,x)\| = 0, \tag{25}$$

system (24) *has a T-periodic solution.*

Mawhin-type result

Corollary 16

Assume A is free from 2-nilpotents and 3-nilpotents. Then, there is $\varepsilon_o > 0$ such that for any h with $\|h(\cdot, x)\|_\infty \leq \varepsilon_o$, system (24) has a T-periodic solution.

5.4. Kinyon-Sagle system via complex structures

Return to the Kinyon-Sagle system

$$\begin{cases} x_0 = \lambda x_0^2 + (x_1^2 + x_2^2) \\ x_1 = -2cx_0x_2 \\ x_2 = 2cx_0x_1, \end{cases} \tag{26}$$

where $c \neq 0$. There is a (non-trivial) bounded solution to (26) iff either of the following holds: (i) $\lambda = \mu = 0$; (ii) $\lambda\mu < 0$. Denote by $A := A(\lambda, \mu, c)$ the algebra associated to (26). In the case (i), the equation $x^3x = -x^2$ has infinitely many solutions of the form $\frac{1}{c}e_0 + x_1e_1 + x_2e_2$ for any $x_1, x_2 \in \mathbb{R}$. Also, $x^2x^2 \equiv x^2x^3 \equiv x^3x^3 \equiv 0$ for all $x \in A$. In the case (ii), any element $x \in A$ satisfies $x^3x^2 = 0$ and the following two algebraically independent equations $i^2i^2 = -i^2$, $j^3j^3 = -j^2j^2$ are soluble in A for $i = \frac{e_1}{\sqrt{-\lambda\mu}}$ and $j = \frac{e_2}{c}\sqrt{-\frac{\lambda}{\mu}}$

Remarks

(i) The above observations indicate an explicit connection between the existence of bounded solutions to system (26) and complex structures in the algebra $A(\lambda, \mu, c)$. However, in general, a three-dimensional Riccati equation amounts to "rank four algebras", i.e. the algebras for which the higher degrees can be expressed via x, x^2, x^3 and the corresponding **linear, quadratic and cubic forms**. Therefore, to study bounded solutions to a Riccati equation occuring in a three-dimensional algebra A, one should look for A-polynomial equations (with real coefficients) containing more than two monomials and naturally generalizing usual real equations having complex roots (possibly, with non-zero real parts).

(ii) Euler and Kasner equations admit an explicit integration. Moreover, all non-trivial solutions to the Euler (resp. Kasner) equation are bounded (resp. unbounded), meaning that they should be considered as the first natural examples where the previous remark is applied.

5.5. Geodesic equations

Return to the geodesic equations on a surface Φ

$$\begin{cases} u'' = -\frac{E_u}{2E}u'^2 - \frac{E_v}{E}u'v' + \frac{G_u}{2E}v'^2 \\ v'' = \frac{E_v}{2E}u'^2 - \frac{G_u}{G}u'v' + \frac{G_v}{2G}v'^2 \end{cases} \tag{27}$$

where Φ is parameterized by

$$r(u,v) = (x(u,v), y(u,v), z(u,v)).$$

and

$$E = E(u,v) := x_u^2 + y_u^2 + z_u^2,$$

$$F = F(u,v) := x_u x_v + y_u y_v + z_u z_v,$$

$$G = G(u,v) := x_v^2 + y_v^2 + z_v^2.$$

Reminder: In contrast to the above examples, where quadratic systems were considered in a fixed linear space, one should consider (27) as a family of systems depending on (u,v).

Clairaut parameterization

For any parameter value (u, v), denote by $A(u, v)$ the algebra associated to the quadratic map in (27)

Clearly, the multiplication in $A(u, v)$ smoothly depends on (u, v).

Question: Can one study geometric properties of the surface Φ by looking at (smooth) deformation of multiplications in $A(u, v)$? If so, what is the role of complex structures in this process?

To be more specific,

Definition 17

A parametrization $r(u, v)$ of Φ is said to be Clairaut in u (resp. Clairaut in v) if $E_v = G_v = 0$ (resp. $E_u = G_u = 0$) for all (u, v).

Clearly, a Clairaut parametrization reduces the 6- parameter family of geodesic equations to the 3-parameter one. On the other hand, it is well-known that studying the six-dimensional space of commutative two-dimensional algebras can be reduced to a two-dimensional family of their isomorphism classes.

Question: Is there any parallelism between these two reductions?

5.6. Surjective quadratic maps

Definition 18

Let $A = (\mathbb{R}^n, \circ)$ be a commutative real algebra.
(i) A is called **regular** if there exists $v \in A$, such that the linear operator defined by $x \to v \circ x$ is invertible. Otherwise, A is called **singular**.
(ii) A is called **unital** if there exits an element $e \in A$ such that $e \circ a = a$ for every $a \in A$.
(iii) A is called **square root closed** if the equation $x^2 = c$ is solvable for any $c \in A$.

Definition 19

Two n-dimensional commutative algebras $A_1 := (\mathbb{R}^n, \circ)$ and $A_2 := (\mathbb{R}^n, *)$ are said to be **isotopic** if there exist $M, L \in GL(n, \mathbb{R})$ such that

$$x \circ y = ML^{-1}\,(Lx * Ly) \qquad (x, y \in \mathbb{R}^n). \tag{28}$$

Remark A is square root closed iff the quadratic map $x \to x^2$ is surjective.

Theorem 20

Let $A = (\mathbb{R}^n, \circ)$ be a commutative algebra.

(i) A is square root closed iff the quadratic map $x \to x^2$ is surjective in A;

(ii) If A is square root closed, then A is regular;

(iii) If A is regular (resp. singular), then any its isotopic image is regular (resp. singular);

(iv) If A is regular, then A is isotopic to a unital algebra;

Proof.

(i) Tautology.

(ii) Assume A is singular. Since

$$x \circ y \equiv \frac{1}{2} J_Q(x) y, \tag{29}$$

where $J_Q(x)$ is the Jacobian matrix of the quadratic map $Q(x) = x \circ x$ at the point $x \in \mathbb{R}^n$, it follows that $\det(J_Q(x))$ must be equal to zero for all $x \in \mathbb{R}^n$. Hence, by the classical Sard-Brown's Theorem, Q cannot be surjective, i.e. A is not square root closed. The contradiction completes the proof.

(iii) Obvious.

(iv) The so-called, Kaplansky's trick. □

Result in 3D

Remark. Theorem 29 reduces the study of surjective quadratic maps to description of regular unital square root closed commutative algebras.

Theorem 21

In order a 3-dimensional commutative regular unital algebra A with unit e to be square root closed, it is necessary and sufficient that the following equations are non-trivially solvable in A:

(i) $e \circ x = x$ *for all* $x \in A$;

(ii) $i^2 = -e$ *for some* $i \in A$;

(iii) $i \circ j = \lambda j$ *for some* $\lambda \in \mathbb{R}$ *and* $j \in A$;

(iv) $k^2 = j$ *for some* $k \in A$;

(v) $l^2 = -j$ *for some* $l \in A$.