Existence and stability of limit cycles in control of anti-lock braking systems with two boundaries via perturbation theory

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Abstract

This paper presents a control logic for anti-lock braking systems (ABS). ABS are by now a standard component in every modern car, preventing the wheels from going into a lock situation where the wheels are fixed by the brake and the stopping distances are greatly prolonged. There are different approaches to such control logics. An ABS design proposed in recent literature controls the wheel’s slip by creating stable limit cycles in the corresponding phase space. This design is modified via an analytical approach that is derived from perturbation theory.

Keywords: Vehicle dynamics, closed-loop switching control, ABS, quarter car model, hybrid systems, limit cycles, non-smooth dynamics, perturbation theory, stability.

1. Introduction

Electronic anti-lock braking systems (ABS) have become a standard for all modern automobiles. They are used to shorten the stopping distance and enhance security. ABS are automated systems that use the principles of threshold braking and cadence braking that were practised by skillful drivers

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with previous generation cars and operate at a much faster rate and with better control than a driver could achieve. They control the braking pressure and by this means the slip of each wheel of a vehicle to prevent locking up - that is ceasing rotation. In the case of locking, the effectiveness of braking and the steerability of the vehicle are significantly reduced. ABS can be modelled as dynamical systems combined with adequate control logics. In the literature various ways are proposed of how to design such controls. Starting from Tanelli et al. [9], we develop an alternative design. Via perturbation theory we analyse the existence of stable limit cycles corresponding to desired behaviour and give an analytical access to the problem. The vehicle’s model and the control logic, we base our approach on, are introduced in Tanelli et al. [9]. Additionally, for instance Johansen et al. [3] and Pasillas-Lépine [8] provide background knowledge about other control logics.

2. Fundamental dynamics

2.1. Introduction of the quarter car model

To design a reasonable control logic, we introduce a model of the vehicle’s dynamics including its wheels. For this purpose, we resort to the quarter car model. It reduces the vehicle’s dynamics to the dynamic of one wheel, neglecting both the more complex interactions between the individual wheels and the changing load on the wheels. Instead, the system’s dynamics are fully described by two variables: the velocity \( v \) of the vehicle itself and, secondly, the angular speed \( \omega \) of the wheel that is analysed. The resulting equations are given by

\[
\begin{align*}
    m \cdot \dot{v} &= -F_x(v, \omega), \\
    J \cdot \dot{\omega} &= r \cdot F_x(v, \omega) - T_b \cdot \text{sgn}(\omega).
\end{align*}
\]

Equations (1) result from basic mechanics, the first equation directly from Newton’s Law, whereas the second equation results from Newton’s Law adapted to angular movement and taking into account the effect of the braking torque \( T_b \geq 0 \). The braking torque always operates in opposite direction to the angular movement \( \omega \), thus resulting in the factor \( \text{sgn}(\omega) \).

We analyse trajectories based at \( v(0) = v_0 > 0, \ \omega(0) = \frac{v(0)}{r} \), corresponding to a vehicle that is moving at velocity \( v_0 \) and without braking. The ABS is switched off at some minimal velocity \( v_{\text{min}} > 0 \). Therefore, we can assume \( v(t) > 0 \) during the entire time.

The second equation contains non-continuous terms giving rise to a switching
surface $\Sigma_\omega$ at $\omega = 0$. Considering $r \cdot F_x > 0$ and $v > 0$, there occur three different situations at $\omega = 0$:

1. For $r \cdot F_x(v, 0) > T_b$, we find $\dot{\omega} > 0$.
2. For $r \cdot F_x(v, 0) = T_b$, we find $\dot{\omega} = 0$.
3. In the case of $r \cdot F_x(v, 0) < T_b$, we gain due to $\text{sgn}(\omega)$

$$\dot{\omega} = \frac{1}{J} \cdot (r \cdot F_x(v, \omega) - T_b) < 0, \quad \text{for } \omega > 0,$$

$$\dot{\omega} = \frac{1}{J} \cdot (r \cdot F_x(v, \omega) + T_b) > 0, \quad \text{for } \omega < 0. \quad (2)$$

Following Filippov a sliding motion given by $\dot{\omega} = 0$ is assumed as long as the trajectory stays in $\Sigma_\omega$.

Considering all three possible cases, for every $v > 0$ the term $\text{sgn}(\omega)$ prevents the derivative of $\omega$ at $\omega = 0$ to become less than 0 and, as expected, braking never results in $\omega(t) < 0$.

The dynamics of the system is non-linear due to the complex dependence of $F_x$ on the state variables $v$ and $\omega$. Therefore, $F_x$ also depends on parameters such as condition of the road, tire and suspension described by a parameter-vector $\theta_r$.\(^1\)

\(^1\)It also depends on the the side-slip angle of the vehicle which we omit assuming the vehicle moving in a straight line.
Definition 1. Instead of $\omega$, the slip $\lambda$ defined by
\[ \lambda := \frac{v - \omega \cdot r}{\max\{\omega \cdot r, v\}}, \quad v > 0. \tag{3} \]
is the more natural variable to base the analysis on and we will substitute the state variable $\omega$ by $\lambda$.

Since we examine a braking situation, the vehicle’s velocity is always larger than the velocity of the wheel at its outer side $v \geq \omega \cdot r$, leading to
\[ \lambda = \frac{v - \omega \cdot r}{v}. \tag{4} \]

We have derived $\omega \geq 0$ and thus gain $\lambda(t) \in [0, 1]$ for $t > 0$. $\lambda = 0$ corresponds to the wheel being as fast as the vehicle, e.g. no braking, whereas $\lambda = 1$ corresponds to $\omega = 0$, e.g. the lock situation, where the wheel does not rotate, but is fully locked by the brakes. Most of the dynamics does not directly depend on $\omega$, but on the ratio between $\omega$ and $v$, e.g. on $\lambda$. Especially, this holds true for the friction coefficient:

Definition 2. Let the friction coefficient $\mu(\lambda; \theta_r)$ be given as a function in dependence on the wheel slip $\lambda$ and the road conditions $\theta_r$
\[ \mu : [0, 1] \times \mathbb{R}^n \rightarrow \mathbb{R}^+. \tag{5} \]

Using $\mu$, the friction force $F_x$ can be put in the form
\[ F_x(\lambda; \theta_r) = F_z \cdot \mu(\lambda; \theta_r) = m \cdot g \cdot \mu(\lambda; \theta_r), \tag{6} \]
where $F_z$ is the constant vertical force at the tire-road contact point generated by gravity and $\mu$ is the friction coefficient. We state three general properties of $\mu(\lambda; \theta_r)$.

1. The coefficient will always be greater or equal zero.
2. In the case of the slip $\lambda$ being zero, the coefficient $\mu$ will be zero too, corresponding to the situation without braking.\(^2\)

\(^2\)We omit here any rolling friction.
3. In all standard situations \( \mu(1; \theta_r) \) is relatively small in comparison to \( \max_{\lambda \in [0, 1]} \mu(\lambda; \theta_r) \). This property motivates the implementation of an anti-lock braking system. It keeps the wheel from locking at \( \lambda = 1 \) where the stopping distance is accordingly prolonged.\(^3\)

These properties correspond to

\[
\begin{align*}
1. & \quad \mu(\lambda; \theta_r) \geq 0, \quad \lambda \in [0, 1], \\
2. & \quad \mu(0; \theta_r) \equiv 0, \\
3. & \quad \mu(1; \theta_r) \ll \max_{\lambda \in [0, 1]} \mu(\lambda; \theta_r).
\end{align*}
\]  

**Definition 3.** \( \mu(\lambda; \theta_r) \) has at least one maximum on \([0, 1]\). Let the one with smallest \( \lambda \) be located at \( \lambda_\mu \in [0, 1] \),

\[
\mu(\lambda_\mu; \theta_r) = \max_{\lambda \in [0, 1]} \mu(\lambda; \theta_r),
\]

In the following, we will resort to a simple and widely used model of the friction coefficient for instance introduced in Kiencke et al. \cite{5} and Burckhardt \cite{2}. In this model, the parameter-vector \( \theta_r \) is three-dimensional

\[
\theta_r = (\theta_{r1}, \theta_{r2}, \theta_{r3}),
\]

\[
\mu(\lambda; \theta_r) = \theta_{r1} \cdot (1 - e^{-\lambda \cdot \theta_{r2}}) - \lambda \cdot \theta_{r3},
\]

and represents different road conditions. We present two configurations corresponding to dry and wet asphalt that are given by

\[
\begin{align*}
\theta_{r_{\text{dry}}} &= (1.11, 23.99, 0.52), \\
\theta_{r_{\text{wet}}} &= (0.687, 33.822, 0.347).
\end{align*}
\]

As long as we examine the system at constant external conditions, we omit the dependence of \( \mu \) on \( \theta_r \).

**Proposition 1.** \( \mu(\lambda) \) as given in (9) is strictly concave and has a unique maximum at \( \lambda = \lambda_\mu \) on \([0, 1]\).

Proof: Trivial.
Next, to replace $\omega$ by $\lambda$, we use the algebraic relation (4). Differentiating leads to

$$\dot{\lambda} = -\frac{r}{v} \cdot \dot{\omega} + \frac{r \cdot \omega}{v^2} \cdot \dot{v},$$

(11)

and by substituting (11) and (6) into (1), we gain

$$\dot{v} = -\frac{F_z}{m} \cdot \mu(\lambda),$$

$$\dot{\lambda} = -\frac{1}{v} \cdot \left(1 - \frac{\lambda}{m} + \frac{r^2}{J}\right) \cdot F_z \cdot \mu(\lambda) + \frac{r}{vJ} \cdot T_b \cdot \text{sgn}(1 - \lambda),$$

(12)

where $v > 0$, $\lambda \geq 0$ as a description of the dynamics of the system. Recall that equations (12) contains the complete information about the effectiveness of the braking maneuver. Optimising the effectiveness corresponds to maximising the decrease of the velocity during a given time-interval $[t_0, t_1]$. The decrease of velocity $\dot{v}$ at a certain time depends (besides some constants)

\[3\] As mentioned in Olson et al. [7], in the special case of packed snow, we find $\mu(1; \theta_r) = \max \mu(\lambda; \theta_r)$ rendering the concept of an ABS useless. In practice, ABS are here manually or automatically switched off.
only on the friction coefficient $\mu(\lambda(t))$. Integrating the first of equations (12) over the time-interval leads to an expression for the decrease $\Delta v$ of $v$:

$$\Delta v = \frac{F_z m}{J m} \int_{t_0}^{t_1} \mu(\lambda(t)) \, dt.$$  

Thus, in the sense specified by the equation above, $\lambda(t)$ should stay close to $\lambda_\mu$ over the braking procedure.

To structure the dynamics, we introduce the function $\psi : [0, 1] \times \mathbb{R}^3 \to \mathbb{R}$ by

$$\psi(\lambda; \theta_r) := \left( \frac{J}{m \cdot r} \cdot (1 - \lambda) + r \right) \cdot F_z \cdot \mu(\lambda; \theta_r),$$

leading to

$$\dot{\lambda} = -\frac{r}{v \cdot J} \cdot [\psi(\lambda) - T_b \cdot \text{sgn}(1 - \lambda)].$$

Similar to $\lambda_\mu$ we define the maximum value of $\psi(\lambda)$ to be located at $\lambda_\psi$.

In table 2 we give an overview of the relevant constants occurring in the quarter car model. In simulations, we will use a specific set of constants corresponding to a certain vehicle. Adapting them to other vehicles does not lead to any qualitative changes.

**Remark 1.** From equation (12), it follows that the dynamic of $\lambda$ becomes infinitively fast for $v \to 0$. We can neglect the situation for small $v$, since ABS are switched off when the velocity has reached a threshold $v_{\min}$ or locking up is accepted.

### 2.2. Change of braking torque

Until now, we have been describing the dynamics of the system affected by a constant braking torque $T_b$. The braking torque is generated by the
driver who exerts the pedal transmitting hydraulic pressure $p_b$ on the brake. We use a static brake-pad friction model leading to the simple connection

$$T_b = r_d \cdot \chi \cdot p_b,$$

(16)

where $r_d$ is the brake disk radius and $\chi$ is the constant friction coefficient. In modern cars a pressure modulator is implemented in the hydraulic system. We consider the following model with two so-called on/off-valves. Basically, this pressure modulator allows to modify the braking pressure in three different ways depending on the state of its two valves, the build valve $v_1$ and the dump valve $v_2$.

- If $v_1$ is opened while $v_2$ remains closed, the pump increases the braking pressure.
- If both valves are closed, the pressure is kept constant.
- If $v_1$ is closed and $v_2$ opened, the pressure decreases.

Naturally, triggering, opening and closing the valves take a certain amount of time. Any control of the valves should not include situations where a valve is opened and closed instantaneously. Thus, we propose a minimal time $t_{\text{min}} > 0$ between two changes of the state of one valve.

The increase and decrease rate of $T_b$ are described by two rate limits $k_1 > 0$ and $k_2 < 0$. Thus, the dynamics of the pressure $T_b$ results in

$$\dot{T}_b = u,$$

(17)
where the choice of the state of the pressure modulator \( u \in \{k_1, 0, k_2\} \) results from the control logic.\(^4\) Combined with equation (15) and the control logic, we find the complete description of the dynamics of the system. The control logic should switch adequately between the different options of \( u \) to result in a stable behaviour of the vehicle.

3. General objectives of ABS control logics

When a braking maneuver is started, we assume the vehicle to move at constant velocity \( v(0) = v_0 \) in a state of non-braking \( T_b(0) = 0 \) and \( \lambda(0) = 0 \). Having introduced a model of the dynamics of the vehicle by equations (15) and (17), we continue by developing general objectives of how to control of the pressure modulation. Various designs have already been proposed such as in Pasillas-Lépine [8]. The state variables, especially the wheel slip \( \lambda \), are difficult to measure or to compute; there exists a variety of different direct and indirect methods of measuring \( \lambda \), each beset with problems\(^5\). Also, the friction coefficient \( \mu(\lambda, t) \) can only be approximated and will change unpredictably according to the road conditions. As key-features of such designs we identify the following objectives:

1. maximising the braking effectiveness corresponding to (13),
2. straightforward implementation,
3. robustness with regard to measure uncertainties and change of road conditions.

4. Results of Tanelli et al.

In Tanelli et al. [9], a design of a control logic for an ABS has been proposed featuring four control phases. The two actuator rates \( k_1 = k \) and \( k_2 = -k \) are assumed to be of equal modulus. The pressure modulator switches between the different choices of \( u \), triggered by the thresholds. Beginning at \( u = k \), if \( T_{b_{\text{max}}} \) is reached \( u \) is switched to 0, then, at \( \lambda_{\text{max}} \), to \( u = -k \) up to the point where \( T_b = T_{b_{\text{min}}} \) where \( u \) is again chosen to be zero.

\(^4\)Since it is physically not possible that the braking pressure becomes negative we find \( T_b \geq 0 \). Thus, to be more precise, \( T_b = u \) is restricted to \( u \in \{0, k_1\} \) at \( T_b = 0 \).

\(^5\)For a list see for instance Ahn et al. [1].
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
<th>Reference Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>(k_1)</td>
<td>change of braking torque</td>
<td>10000</td>
</tr>
<tr>
<td>(k_2)</td>
<td>change of braking torque</td>
<td>10000</td>
</tr>
<tr>
<td>(T_{b_{\text{min}}})</td>
<td>minimal torque</td>
<td>1000</td>
</tr>
<tr>
<td>(T_{b_{\text{max}}})</td>
<td>maximal torque</td>
<td>1600</td>
</tr>
<tr>
<td>(\lambda_{\text{min}})</td>
<td>minimal slip</td>
<td>0.09</td>
</tr>
<tr>
<td>(\lambda_{\text{max}})</td>
<td>maximal slip</td>
<td>0.35</td>
</tr>
</tbody>
</table>

Table 3: Reference values used in Tanelli et al. [9].

and finally at \(\lambda_{\text{min}}\) to \(u = k\).

This leads to the sequence \(k \rightarrow 0 \rightarrow -k \rightarrow 0 \rightarrow k\) and the following switching manifolds:

\[
\begin{align*}
\tilde{\Sigma}_0 &= \{ (\lambda, T_b) : H_0(\lambda, T_b) := T_b - T_{b_{\text{max}}} = 0 \}, \\
\tilde{\Sigma}_1 &= \{ (\lambda, T_b) : H_1(\lambda, T_b) := \lambda - \lambda_{\text{max}} = 0 \}, \\
\tilde{\Sigma}_2 &= \{ (\lambda, T_b) : H_2(\lambda, T_b) := T_b - T_{b_{\text{min}}} = 0 \}, \\
\tilde{\Sigma}_3 &= \{ (\lambda, T_b) : H_3(\lambda, T_b) := \lambda - \lambda_{\text{min}} = 0 \}. \\
\end{align*}
\]

The control logic is implemented using the values given in table 3. We note that over one cycle \(\lambda\) spreads at least between \([0.09, 0.35]\). In [9], the dynamics given in (23) are slightly modified and then used to examine the control logic. The function \(\mu\) and the four threshold values are subject to three conditions denoted (i) to (iii)\(^6\) which assure that the switching logic generates a stable limit cycle in the phase space as shown in figure (4). The closed orbit evolves in a nearly rectangular way along the four switching surfaces. Their properties can easily be analysed geometrically. For each of the four control phases the corresponding attractor can be determined containing the points of the phase space whose trajectories converge to the closed orbit. By this means, one can derive that to a certain extend the observed limit cycle

1. is structurally stable with respect to measurement uncertainties and change of road conditions, and
2. is robust to variation of the actuator rate-limit \(k\).

\(^6\)See [9], p.667
On the other side, this control design gives rise to several problems.

1. The usage of \( \lambda \) as a control variable enforces large safety margins since the methods of measurement of this parameter are either too complex to implement or rather imprecise.

2. To ensure stability of the cycle the boundary values \( \lambda_{\text{min}} \) and \( \lambda_{\text{max}} \) are required to be considerably distant.\(^7\) As can be seen in figure 4, the closed orbit even includes parts with \( \lambda(t) < \lambda_{\text{Min}} \) and \( \lambda(t) > \lambda_{\text{Max}} \). Thus, \( \lambda \) does not stay in the neighbourhood of the maximum \( \lambda_{\mu} \), impairing the effectiveness of the braking procedure.

3. The design is complex, since four thresholds that depend on two parameters need to be taken into consideration and three conditions that depend additionally on \( \mu(\lambda) \) need to be met to ensure stability.

5. Alternative simplified control logic

We propose a design of a control logic that only includes two control phases and two switching surfaces and avoids usage of \( \lambda \) as a control parameter. Additionally, the state of the valves corresponding to \( u = 0 \) is not used. The two switching surfaces are given by

\[
\Sigma_0 = \{ (\lambda, T_b) : H_0(\lambda, T_b) := T_b - T_{b_{\text{min}}} = 0 \}; \\
\Sigma_1 = \{ (\lambda, T_b) : H_1(\lambda, T_b) := T_b - T_{b_{\text{max}}} = 0 \}. \tag{19}
\]

Whenever the trajectory hits the upper surface \( \Sigma_1 \) corresponding to \( T_b = T_{b_{\text{max}}} \), the change of the braking torque \( \dot{T}_b = u \) shall switch from \( u = k_1 > 0 \)

\(^7\)Compare to table 3.
to $u = k_2 < 0$. On the contrary, as soon as the lower boundary at $T_{b_{\text{min}}}$ is hit, it shall switch from $u = k_2$ back to $u = k_1$. We define the distance between the two surfaces by

$$\Delta T_b := T_{b_{\text{max}}} - T_{b_{\text{min}}} > 0.$$  \hfill (20)

With $T_c$ being the period of one cycle, we find

$$T_c = \frac{\Delta T_b}{k_1} + \frac{\Delta T_b}{k_2},$$  \hfill (21)

which is, considering the usual choice of constants, about $\frac{1}{10} s^{-1}$. Due to (13) we find

$$\Delta v = \frac{F_z}{m} \cdot \int_0^{T_c} \mu(\lambda(t)) \, dt \leq \frac{F_z}{m} \cdot T_c \cdot \mu(\lambda),$$  \hfill (22)

which results at a value around $\Delta v = 1 \frac{m}{s}$. Thus, regarding the dynamics of $\lambda$ in (12), on a short time scale the change of $v$ is insignificantly small. We consider it as a slowly varying variable and neglect the first equation of (12) leading to $v_{\text{constant}}$. Following (15) and (17) the system’s dynamics is reduced to

$$\dot{\lambda} = -\frac{r}{v} \cdot J \cdot [\psi(\lambda) - T_b \cdot \text{sgn}(1 - \lambda)],$$

$$\dot{T}_b = u,$$  \hfill (23)

where $u \in \{k_1, k_2\}$.

As mentioned, the time between the two changes of the state $u$ should exceed a minimal time $t_{\text{min}} > 0$. This results directly in a minimal distance between the thresholds of the control that is given by

$$\Delta T_b \geq t_{\text{min}} \cdot \max \{\|k_1\|, \|k_2\|\}.$$  \hfill (24)

We note that, due to the term $\text{sgn}(1 - \lambda)$ occurring in the first equation of (23), the situation at $\lambda = 1$ is non-smooth and requires further investigation. The behaviour depends strictly on the relation between the variable $T_b$ and the constants. For $r \cdot F_z \cdot \mu(1) > T_b$ we find $\dot{\lambda} < 0$ and for $T_b \geq r \cdot F_z \cdot \mu(1)$, we

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8Nevertheless, the control logic is needed to work at any occurring velocity.
Figure 5: A trajectory created by the control logic at constant $v$.

Figure 6: Illustration of crossing and sliding regions of the set $\Sigma_\lambda$.

\[ T_b \]

\[ J \]
\[ \Rightarrow J \subset \Sigma_\lambda^S \]

\[ I \]
\[ \Rightarrow I \subset \Sigma_\lambda^C \]

find $\dot{\lambda} = 0$. Following Filippov, the set $\Sigma_\lambda = \{(\lambda, T_b) | \lambda = 1\}$ is a switching surface devided into the set of direct crossing

\[ \Sigma_\lambda^C = \{(\lambda, T_b) | \lambda = 1, r \cdot F_z \cdot \mu(1) > T_b\} \subset \Sigma_\lambda, \quad (25) \]

and the set of sliding

\[ \Sigma_\lambda^S = \{(\lambda, T_b) | \lambda = 1, T_b \geq r \cdot F_z \cdot \mu(1)\} \subset \Sigma_\lambda. \quad (26) \]

Combined, we find $\dot{\lambda} \leq 0$ for all states at $\lambda = 1$.

As far as the situation at $\lambda = 0$ is concerned, we have $\dot{\lambda} \geq 0$ at $\lambda = 0$. We conclude that the set $\{(\lambda, T_b) | \lambda \in [0, 1]\}$ is positively invariant.

We continue by examining the relation between $\mu$ and $\psi$.

**Proposition 2.** Similar to $\mu(\lambda)$ and $\lambda_\mu$ as given in definition 3, it holds that $\psi(\lambda)$ is a strictly concave function and there exists a unique value $\lambda_\psi$

\[ \lambda_\psi \in (0, \lambda_\mu), \quad (27) \]
so that $\psi(\lambda_\psi)$ is maximal. Furthermore, we find

$$
\begin{align*}
\psi'(\lambda) &> 0, \quad \lambda < \lambda_\psi, \\
\psi'(\lambda) &< 0, \quad \lambda > \lambda_\psi.
\end{align*}
$$

(28)

Proof: Trivial.

Remark 2. As follows from (14), $\psi(\lambda)$ equals $\mu(\lambda)$ being scaled by the constant factor $F_z$ and by the sum of $\frac{J}{m \cdot r} \cdot (1 - \lambda)$ and $r$. The first term is maximal for $\lambda = 0$, but its modulus will be comparably small for all proper choices of the parameters

$$
\left\| \frac{J}{m \cdot r} \right\| \ll 1.
$$

(29)

Thus, $\psi(\lambda)$ is similar to $F_z \cdot r \cdot \mu(\lambda)$, for instance concerning the location of their maxima

$$
\lambda_\psi \approx \lambda_\mu,
$$

(30)

which can also be seen in figure 7 and table 4.
6. Analysis via homotopy

In this section, we analyse the proposed ABS design for the quarter car model by perturbation theory. Similar to Tanelli et al. [9], the proposed logic should result in asymptotically stable limit cycles. The perturbation approach will provide an excellent theoretical insight into the dynamics of the ABS and the effects of the different parameters. In the first step we assume constant road conditions and constant velocity and analyse the intersections of the limit cycles resulting from the control logic with the lower threshold. In a second step we extend our focus to the entire orbit.

6.1. Perturbation method

The perturbation approach as derived in the appendix provides a possibility to detect closed trajectories in the ABS system. As a perturbation method it can only give statements that have to be verified by other means. Nevertheless it provides an analytical access to the problem that makes it notably easier to access the influence of different parameters. The main idea consists in embedding the system by introducing an artificial parameter $\epsilon$ such that

\[
\dot{\lambda} = \epsilon \cdot f(\lambda, T_b),
\]
\[
\dot{T}_b = u,
\]

where $f(\lambda, T_b) = -\frac{v}{\psi(\lambda) - T_b \cdot \text{sgn}(1 - \lambda)}$ and $u$ is chosen by the introduced control logic. For $\epsilon = 0$ the system has a particular easy configuration, whereas for $\epsilon = 1$ we obtain the original system (23).

We consider $\Sigma_0$ as defined in (19) as a Poincaré section and introduce a Poincaré map $P(\lambda, \epsilon) : \Sigma_0 \rightarrow \Sigma_0$ by

\[
P(\lambda, \epsilon) = \lambda (T_c, \lambda, \epsilon).
\]

Thus, for each system the function $P(\lambda, \epsilon)$ maps points $(\lambda_0, T_{b_{\text{min}}}) \in \Sigma_0$ to the corresponding returning points in $\Sigma_0$ of the trajectory, taking into account the two switchings of $u = \dot{T}_b$ at $T_{b_{\text{max}}}$ and $T_{b_{\text{min}}}$, respectively.

By our approach, we determine the location of fixed points of $P$ for $\epsilon > 0$ which are obtained as a function $F(\epsilon) : [0, \epsilon_{\text{max}}) \rightarrow \Sigma_0$, i.e. satisfying

\[
P(F(\epsilon), \epsilon) = F(\epsilon), \quad \text{for all } 0 \leq \epsilon < \epsilon_{\text{max}}.
\]
The resulting $F$ will be of the form $F(\epsilon) = L_0 + L(\epsilon) \cdot \epsilon$. At least for small $\epsilon$ we can obtain a first order approximation $L_1$:

$$F(\epsilon) = L_0 + L_1 \cdot \epsilon + O(\epsilon^2).$$

(34)

With some preconditions fulfilled, the existence of the function $F(\epsilon)$, stability of the corresponding limit cycles and the first order approximation are determined by three conditions depending on the Poincaré map: Necessary for the existence we find

$$D_\epsilon P(\lambda_0, 0) = 0,$$

(35)

for some $\lambda_0$. Sufficient to its existence is, besides (35), that

$$\text{det}(D_\lambda D_\epsilon P(\lambda_0, 0)) \neq 0.$$

(36)

The stability of the corresponding trajectories is at least for small $\epsilon$ determined by the eigenvalues of $D_\lambda D_\epsilon P(L_0, 0)$. If they have negative real parts, the trajectory is asymptotically stable. The first order approximation is given by the lengthy but computable expression (A.35).

**Definition 4.** In the expression (34) of $F(\epsilon)$, we directly find $F(0) = L_0$. At $L_0$, the fixed points $F(\epsilon)$ bifurcate from the variety of fixed points at $\epsilon = 0$ where every point is a fixed point. We refer to $L_0$ as the point of origin or simply origin.

6.2. Analysis of fixed points

We define the average between the two thresholds:

$$T_{bavg} := \frac{T_{bmax} + T_{bmin}}{2}.$$  

(37)

Due to (A.33), we find

$$D_\epsilon P(\lambda_0, 0) = -\left(\frac{1}{k_1} + \frac{1}{k_2}\right) \cdot \frac{r}{v} \cdot \Delta T_b \cdot \left[\psi(\lambda_0) - T_{bavg}\right],$$

(38)

and thus the necessary condition $D_\epsilon P(\lambda_0, 0) = 0$ is equivalent to

$$\psi(\lambda_0) = T_{bavg}.$$  

(39)
Therefore, for $\psi(\lambda_0)$ equal to the average of the two boundary values, $\lambda_0$ could be the origin of a function of fixed points $F(\epsilon)$. Consequently, the boundary values $T_{b_{\min}}$ and $T_{b_{\max}}$ need to be chosen in such a way that

$$T_{b_{\text{avg}}} \in \{\psi(\lambda), \lambda \in [0,1]\}.$$  

(40)

Assuming, we have found such $\lambda_0$, we examine the sufficient condition and the stability via $D_{\lambda}D_{\epsilon}P$ and gain due to (A.34)

$$D_{\lambda}D_{\epsilon}P(\lambda_0,0) = -\left(\frac{1}{k_1} + \frac{1}{k_2}\right) \frac{r}{\psi \cdot J} \cdot \psi'(\lambda_0) \cdot \Delta T_b.$$  

(41)

Due to the positiveness of all constants, we conclude that $\lambda_0$ with $\psi(\lambda_0) = T_{b_{\text{avg}}}$ is an origin of asymptotically stable fixed points if and only if $\psi'(\lambda_0) > 0$. Since

$$\psi'(\lambda) > 0, \quad \text{if and only if} \quad \lambda \in [0,\lambda_\psi),$$  

(42)

and

$$\psi([0,\lambda_\psi)) = [0,\psi(\lambda_\psi)),$$  

(43)

we conclude that an asymptotically stable limit cycle exists for small $\epsilon$ if the boundaries $T_{b_{\max}}$ and $T_{b_{\min}}$ are chosen such that

$$T_{b_{\text{avg}}} < \psi(\lambda_\psi) = \max_{\lambda \in [0,1]} \psi(\lambda).$$  

(44)

As a matter of course, additionally to the perturbation analysis, we still need to examine the situation at $\epsilon = 1$ closer.

Furthermore, it is indicated to evaluate if the generated limit cycles also meet the requirements of a desired braking maneuver given in section 3.

Next, we examine the special situation at $\lambda = 1$. For

$$r \cdot F_z \cdot \mu(1) < T_{b_{\text{avg}}},$$  

(45)

different types of closed orbits exist depending on $T_{b_{\min}}$ and $T_{b_{\max}}$:
• For $T_{b_{min}} \geq r \cdot F_z \cdot \mu(1)$, trajectories stick entirely to the switching surface $\Sigma_{\lambda}$ and are stable limit cycles. The wheel is kept locked throughout the whole orbit.

• For $T_{b_{min}} < r \cdot F_z \cdot \mu(1)$, more complex trajectories develop. The orbit sticks on its way downwards to the switching surface $\Sigma_{\lambda}$ until it leaves $\Sigma_{\lambda}$ at $T_b(t) = r \cdot F_z \cdot \mu(1)$, but returns eventually. Such stick-slip phenomena occur in many systems with friction. Figure 8 depicts an exemplary trajectory.

• In the case $T_{b_{avg}} > \psi(\lambda_{\psi})$ there are no stable limit cycles besides the one at $\lambda = 1$ and the ABS always locks the wheels.

In figure 9, we present the corresponding bifurcation diagram of fixed points for dry road conditions in dependence on $T_{b_{avg}}$. The curve marked by circles results from (40) and (44) respectively and therefore from the function $\psi$: Values of $\lambda$ where $\psi(\lambda)$ has positive slope are stable, whereas negative slope leads to unstable limit cycles. At $T_{b_{avg}} = \psi(\lambda_{\psi})$ a classical saddle-node bifurcation occurs.

Furthermore, we can assess the attractor of each stable fixed point of the Poincaré map. As also visualised in figure 9, the attractor of each fixed point $\lambda_0$ with $\psi(\lambda_0) = T_{b_{avg}}$ is bounded by 0 on the left hand side and either 1 or the location of the unstable fixed point on the right hand side. Thus, for increasing $T_{b_{avg}}$ the attractor contracts to $[0, \lambda_{\psi}]$ at $T_{b_{avg}} = \psi(\lambda_{\psi})$ and stability weakens until it is lost completely in the course of the bifurcation.
Remark 3. In an additional step, perturbations of the increase and decrease rate of the braking torque $k_1$ and $k_2$ can be analysed by introducing functions $g_1$ and $g_2$ in (A.28) and A.29). It follows that closed orbits are robust to perturbations of the braking torque for small $\epsilon$. Perturbations of the braking torque are, for instance, expected to occur due to mechanical inaccuracies of the pressure modulator (see Köppen [6]).

6.3. First order approximation

The perturbation method provides us with a first order approximation $L_1$ of the location of the fixed points.

$$F(\epsilon) = L_0 + L_1 \cdot \epsilon + O(\epsilon^2).$$

(46)

$L_1$ is computed via (A.35). Without any perturbations of the torque increase and decrease rate of $T_b$, we find

$$L_1 = \frac{1}{12} \frac{r}{v \cdot J} \left( \frac{1}{k_1} + \frac{1}{k_2} \right)^{-1} \left( \frac{1}{k_1^2} - \frac{1}{k_2^2} \right) \left( T_{b_{\text{max}}} - T_{b_{\text{min}}} \right)^2.$$ (47)

We note

$$L_1 = 0, \quad \text{for} \quad k_1 = k_2,$$

$$L_1 \neq 0, \quad \text{for} \quad k_1 \neq k_2. \quad (48)$$

Thus, by varying $k_1$ and $k_2$, we can manipulate the first order approximation $L_1$ and, for instance, try to relocate the fixed points $F(\epsilon)$ at $\epsilon = 1$ closer to the optimal location $\lambda_\mu$. A visualisation of the effect of different choices of $k_1$ and $k_2$ on the first order approximation is given in figure 10. First, we realise that, in contrast to the location of the origin $L_0$, the first order approximation $L_1$ depends on the distance between $T_{b_{\text{min}}}$ and $T_{b_{\text{max}}}$ and on the velocity of the vehicle $v$. (47) gives the first order approximation belonging to a certain origin $L_0$ that itself results from a certain choice of $T_{b_{\text{min}}}$ and $T_{b_{\text{max}}}$. So, due to (39), constant average needs to be maintained when adjusting $L_1$. Then, by increasing or decreasing the distance $\Delta T_b$ we gain different $L_1$ as shown in figure 10. The manipulation of the first order approximation has also a

---

9 Larger perturbations influence the first order approximation and the orbits up to the point where their stability or existence at $\epsilon = 1$ are lost.

10 One can with a little more effort and due to (A.35) also estimate the error from perturbations of $k_1$ and $k_2$ to $L_1$. 

19
substantial impact on the fixed point $F(1)$ and moves indeed the intersection of the orbit with the lower threshold. The entire orbit however is not shifted significantly and the effect on the braking efficiency is near to zero.

6.4. Summary of results for fixed points

As shown in (44), the average between the two boundary values is constitutive for the origin $L_0 = \lambda_0$ of the closed orbits. As this relationship is simple it is easy to analyse. It does not depend on $v$ and, until now, there is no effect of choosing $T_{b_{\mathrm{min}}}$ and $T_{b_{\mathrm{max}}}$ either close or distant to each other as long as their average is not varied. The analysis via the perturbation method does not only provides results for a certain set of parameters but can also show the effects of changes in parameters.

On the other side, as far as the ABS is concerned, our options of manipulating the location of the fixed points are limited: Although parameters $T_{b_{\mathrm{min}}}$, $T_{b_{\mathrm{max}}}$, $k_1$, $k_2$ can be adjusted, only the average $T_{b_{\mathrm{avb}}}$ affects the dynamics significantly.

Furthermore, due to (27) and (28), a conflict of interest arises:

- On the one hand, the wheel needs to be prevented from going into the lock situation. Instead it should evolve into an asymptotically stable orbit where $\lambda(t) \ll 1$.
- On the other hand, during the cycle the value of $\lambda(t)$ should stay as close as possible to $\lambda_\mu$ to maximise the braking efficiency.

---

$^{11} v = 30 \frac{m}{s}$, $T_{b_{\mathrm{min}}} = 800$, $T_{b_{\mathrm{max}}} = 1500$
As we have seen, at $\lambda = \lambda_\mu$ the derivative $\psi'(\lambda_\mu)$ is always negative and, therefore, for any $T_{b_{\text{min}}}$ and $T_{b_{\text{max}}}$ the point $\lambda_0 = \lambda_\mu$ itself cannot be the origin of asymptotically stable fixed points.

Next we draw attention to the orbit itself.

7. Analysis of entire orbits

7.1. Analysis of entire orbits at constant velocity and constant road conditions

Until now, we have only been examining the fixed points at $\Sigma_0$, e.g. the starting points of the closed orbits at $T_{b_{\text{min}}}$, but not the full orbits. Nevertheless, for an analysis of the ABS design, the entire trajectory needs to be taken into account, especially to assess the braking-effectiveness of the control logic. We start by assuming constant velocity and road condition and then discuss how the results are affected if velocity and conditions are variable.

We begin by carrying out a simulation to demonstrate the behaviour of the switching logic. At constant, dry road conditions, we find $\max_{\lambda \in [0,1]} (\psi(\lambda)) = \psi(0.1619) \approx 1208$. The valve rates are chosen to be $k_1 = k_2 = 10000$ resulting in $L_1 = 0$. As far as the boundaries are concerned, we select $T_{b_{\text{min}}} = 800$ and $T_{b_{\text{max}}} = 1500$, expecting the fixed points to rise from $\lambda_0 \approx 0.0979$.

Given this configuration, we gain the result depicted in figure 11 and figure 12. Notice that the deviation of the computed fixed points to their first order approximation is relatively small. For instance, at $\epsilon = 1$ we find

![Fixed Points and First Order Approximation](image-url)
$$\Delta \lambda = F(1) - (L_0 + L_1) \approx 0.0015.$$ We additionally note that the trajectories move along closed orbits whose $\lambda$-values are located left and right from the intersection point $F(1)$. It is important to observe that the variation of $\lambda$ within the cycle spreads over a considerable area, in this case from 0.063 to 0.18.

In Tanelli et al. [9], for both $k_1$ and $k_2$ values of 10000 are used. From (47), it becomes clear that in our design a choice of $k_1 < k_2$, for instance $k_1 = 8000$, $k_2 = 12000$, leads to positive $L_1$. In figure 13 we show corresponding closed orbits at $\epsilon = 1$. As already seen in simulations and also

\begin{itemize}
  \item[a)] $k_1 = k_2 = 10000$, $T_{b_{\text{min}}} = 1000$, $T_{b_{\text{max}}} = 1300$;
  \item[b)] $k_1 = k_2 = 10000$, $T_{b_{\text{min}}} = 800$, $T_{b_{\text{max}}} = 1500$;
  \item[c)] $k_1 = 8000$, $k_2 = 12000$, $T_{b_{\text{min}}} = 1000$, $T_{b_{\text{max}}} = 1300$;
  \item[d)] $k_1 = 8000$, $k_2 = 12000$, $T_{b_{\text{min}}} = 800$, $T_{b_{\text{max}}} = 1500$
\end{itemize}

Figure 12: Numerical simulation of the closed orbit for $\epsilon = 1$.

Figure 13: Orbits at $\epsilon = 1$ for different $\Delta T_b$ (horizontal) and different $k_1, k_2$ (vertical).\textsuperscript{11}
in figure 13, the compliance at $\epsilon = 1$ between the first order approximation $L_0 + L_1 \cdot 1$ and the actual fixed point $F(1)$ is good. Additionally, it is visualised that $L_1$ increases with growing $\Delta T_b$.

Whereas at small $\Delta T_b$ the variation of $\lambda$ over the closed cycle is comparably small, it increases with growing $\Delta T_b$ as the entire orbit blows up. Predictions of the location of the fixed point and its stability become more unreliable as the average of the two boundary values approach $\max_{\lambda \in [0,1]} \psi(\lambda)$.

7.2. Effects of decreasing velocity

Up to now, we have considered the velocity of the vehicle $v$ to be constant which is adequate on the small timescale of one closed orbit. During the entire braking procedure however, the decrease of $v$ has to be taken into consideration. Due to equations (23), the velocity affects the dynamics of $\lambda$ simply as multiplication with $\frac{1}{v}$. Thus, for decreasing $v$ the dynamics of $\lambda$ become faster and the limit cycle itself blows up. In doing so it reduces the braking efficiency. To level this effects, the distance $\Delta T_b$ between the two boundary threshold or the average $T_{b_{avg}}$ could be steadily diminished over the braking procedure up to the minimal distance $\Delta T_b$.

Simulations for decreasing velocity in figure 14 demonstrate the mentioned effects.

We note that, as one would expect, the dynamics become infinitively fast as $v \to 0$. Thus, the limit cycle looses its stability completely at some velocity $v_{instab}$ where $\lambda$ increases to 1 as the wheel goes into lock. This behaviour is

\[ k_1 = k_2 = 10000, \quad T_{b_{min}} = 1000, \quad T_{b_{max}} = 1300 \]
naturally expected and since $v_{\text{instab}}$ is rather small of no further interest. We depict an exemplary loss of stability at small velocity in figure 15.

7.3. Adapting to non-constant road conditions

In this section we highlight the main problem of our control design. Until now, we have been assuming known and constant road conditions resulting in time-independent functions $\mu(\lambda)$ and $\psi(\lambda)$. In this case, we can by the means derived in the previous sections choose appropriate values for $T_{b_{\text{min}}}$, $T_{b_{\text{max}}}$ and $k_1$, $k_2$, leading to good braking performance. In case of unknown and changing road conditions, the choice of the parameters becomes more complicated.

We assume we cannot adapt the values on-line to changing road conditions. Then, we need to choose the values in advance in a way that even in the worst case a stable orbit exists and the wheels are always prevented from going into lock. Unfortunately, this can only be achieved at the expense of the braking effectiveness at other road conditions.

Let us, for instance, consider roads that have sections of dry and wet asphalt and whose arrangements are unknown to us. To assure stability for every situation, the average of $T_{b_{\text{min}}}$ and $T_{b_{\text{max}}}$ needs to be chosen appropriately such that

$$T_{b_{\text{avg}}} \leq \min_{t \geq 0} \max_{\lambda \in [0,1]} \psi(\lambda; \theta_r(t)).$$  \hspace{1cm} (49)

Referring to the maxima for dry and wet asphalt given in table 4, it follows $T_{b_{\text{avg}}} \leq 764$. While at the sections of wet asphalt we gain relatively good braking performance, the choice of the boundary values obviously leads to poor effectiveness on dry asphalt. In this case, the origin of fixed points computes $L_0 = 0.0368$ which is significantly distant to $\lambda_\mu = 0.1641$. We
find \( \mu(L_0) = 0.682 \) and \( \mu(\lambda_\mu) = 1.003 \) and thus a braking procedure on dry asphalt with parameters optimised to wet asphalt is expected to take up to 50\% longer time compared to the optimal setting. Accordingly, also the braking distance is greatly prolonged. Thus, this design needs as well to be extended to adapt to different road conditions. Next we show a basic approach that is inherently included in the ABS design:

We assume we can adapt our choices of \( T_{b_{\text{min}}} \) and \( T_{b_{\text{max}}} \) on-line. Thus, the control logic should adapt the thresholds appropriately, for instance to an estimation of \( \lambda \) or \( \mu \). Whenever \( \max \mu \) was moving below a stable value with regard to \( T_{b_{\text{avg}}} \), either \( T_{b_{\text{avg}}} \) would either directly be readjusted or stability would be shortly lost and \( \lambda \) would start increasing. Then, the system could intercept at a certain value of \( \lambda \), for instance \( \lambda = 0.5 \), and as a countermeasure decrease the boundary values and their average \( T_{b_{\text{avg}}} \) to a stable value.

For now, we conclude that non-constant road conditions cannot be addressed by the perturbation method alone and impose challenges that cannot be solved by the design in its simplest form.

8. Conclusion

In this paper we have presented a design of an anti-lock braking system. The design stands out for its simplicity and easy scalability. Its good braking efficiency can be improved by introducing additional control logics. To analyse the design the usage of perturbation theory provides an analytical access and deeper understanding of the dynamics and effects of parameter changes.

Appendix A. Perturbation method

We first state the applied perturbation approach in a generalised form for Poincaré Maps with specific properties. Then we specify the resulting lemmas to the ABS system and finally give computable formulae that are used in the analysis. All derivations are given in full particulars in Köppen [6].

Appendix A.1. Introduction

Let there be a perturbed dynamical system given by

\[
\dot{x} = f(x, \epsilon), \quad x \in \mathbb{R}^n, \quad \epsilon \in [0, \epsilon_{\text{def}}),
\]  

(A.1)
where \( f : \mathbb{R}^n \times [0, \epsilon_{\text{def}}) \to \mathbb{R}^n \) is a twice continuously differentiable function. Let there be a \((n - 1)\)-dimensional Poincaré section \( \Sigma_0 \subset \mathbb{R}^n \) and a corresponding Poincaré map \( P(x, \epsilon) \) such that
\[
P(\xi, 0) = \xi, \quad \text{for all } \xi \in \Sigma_0, \tag{*}
\]
and \( P \) is a three times continuously differentiable function at \( \epsilon = 0 \). \( P \) may not be defined for all combinations \( \xi \in \Sigma_0 \) and \( 0 \leq \epsilon < \epsilon_{\text{def}} \). Therefore, we denote the domain of \( P \), e.g. those combinations that have a first return, by \( \Lambda \subset \mathbb{R}^n \times [0, \epsilon_{\text{def}}) \) and consider the Poincaré map
\[
P(\xi, \epsilon) : \Lambda \to \Sigma_0. \tag{A.2}
\]
Due to property \((*)\), we find \( \Sigma_0 \times \{0\} \subset \Lambda \). Moreover, we assume that there exists an open neighbourhood of \( \Sigma_0 \times \{0\} \) in \( \Lambda \) so that derivatives of \( P \) at \( \epsilon = 0 \) are well defined.

For convenience, we assume that \( \Sigma_0 \) is not only a hypersurface but a hyperplane and \( 0 \in \Sigma_0 \).\(^{13}\)

At \( \epsilon = 0 \) every point \( \xi \in \Sigma_0 \) is a fixed point of \( P(\xi, \epsilon) \). We would like to find \( 0 < \epsilon_{\text{max}} < \epsilon_{\text{def}} \) and a continuously differentiable function
\[
F : [0, \epsilon_{\text{max}}) \to \Sigma_0, \tag{A.3}
\]
such that
\[
F(\epsilon) = P(F(\epsilon), \epsilon). \tag{A.4}
\]
The description of \( F(\epsilon) \) shall be given by
\[
F(\epsilon) = \xi_0 + \xi(\epsilon) \cdot \epsilon = \xi_0 + \xi_1 \cdot \epsilon + O(\epsilon^2), \tag{A.5}
\]
where \( \xi_0 \in \Sigma_0 \) and \( \xi : [0, \epsilon_{\text{max}}) \to \mathbb{R}^n \) is a continuous function. Furthermore, we predict the stability of the fixed points, e.g. whether they are asymptotically stable around \( \epsilon = 0 \). In this case, we show the existence of a value \( 0 < \epsilon_{\text{stab}} \leq \epsilon_{\text{max}} \), up to which the corresponding trajectories are at least asymptotically stable.

\(^{13}\)By this restriction, we will stay within the manifold when adding two vectors \( u, v \in \Sigma_0 \), simplifying the calculations. Since every Poincaré section is as a hypersurface locally equivalent to \( \mathbb{R}^{n-1} \) and our analysis is operating locally, we do not lose any generality by this confinement.
Definition 5. In the expression (A.5), we directly find $F(0) = \xi_0$. \(\xi_0\) is the bifurcation point at \(\epsilon = 0\). We refer to \(\xi_0\) as the point of origin or simply origin. At \(\xi_0\), the fixed points \(F(\epsilon)\) bifurcate from the variety of fixed points at \(\epsilon = 0\) where every point is a fixed point. The point \(\xi_0\) can be exactly calculated by the perturbation approach in contrast to \(F(1)\).

Appendix A.2. Perturbation method for Poincaré map

Lemma 1. Assume there is a three times continuously differentiable Poincaré map \(P(\xi, \epsilon) : \Lambda \rightarrow \Sigma_0\) such that

\[
P(\xi, 0) \equiv \xi, \quad \text{for all } \xi \in \Sigma_0. \quad (\ast)
\]

Secondly, assume there exists \(\xi_0 \in \Sigma_0, 0 < \epsilon_{\text{max}} < \epsilon_{\text{def}}\) and a continuous function \(\xi(\epsilon) : [0, \epsilon_{\text{max}}) \rightarrow \mathbb{R}^n\) such that

\[
P(\xi_0 + \xi(\epsilon) \cdot \epsilon, \epsilon) \equiv \xi_0 + \xi(\epsilon) \cdot \epsilon = F(\epsilon), \quad (A.6)
\]

for all \(0 \leq \epsilon < \epsilon_{\text{max}}\). Then, it holds that

\[
D_\epsilon P(\xi_0, 0) = 0. \quad (A.7)
\]

Proof sketch:
Expand \(P_\epsilon(\xi, \epsilon)\) in a Taylor series in \(\epsilon = 0\) in the second argument to second order. Use \((\ast)\) to simplify the derived equations and devide by \(\epsilon \neq 0\). Then take the limes \(\epsilon \rightarrow 0\).

Now after postulating the existence of \(F(\epsilon)\) to gain necessary condition \((A.7)\), we obtain conditions that are sufficient to the local existence of the seeked function \(F(\epsilon)\).

Lemma 2. Let there be a three times continuously differentiable Poincaré map \(P(\xi, \epsilon) : \Lambda \rightarrow \Sigma_0\) with the following property:

\[
P(\xi, 0) \equiv \xi, \quad \text{for all } \xi \in \Sigma_0. \quad (\ast)
\]

Assume there exists \(\xi_0 \in \Sigma_0\) such that the necessary condition of theorem 1 holds

\[
D_\epsilon P(\xi_0, 0) = 0, \quad (A.8)
\]
and further assume
\[ \det(D_\xi D_\epsilon P(\xi_0, 0)) \neq 0. \] (A.9)
Then, there exists \( 0 < \epsilon_{\text{max}} < \epsilon_{\text{def}} \) and a continuous function \( \xi(\epsilon) : [0, \epsilon_{\text{max}}) \rightarrow \mathbb{R}^n \) such that
\[ P(\xi_0 + \xi(\epsilon) \cdot \epsilon, \epsilon) \equiv \xi_0 + \xi(\epsilon) \cdot \epsilon, \] (A.10)
for all \( 0 \leq \epsilon < \epsilon_{\text{max}} \). Furthermore, there exists a unique \( \xi_1 \in \mathbb{R}^n \),
\[ \xi_1 = -\frac{1}{2} \cdot (D_\xi D_\epsilon P(\xi_0, 0))^{-1} \cdot D_\epsilon D_\xi P(\xi_0, 0), \] (A.11)
such that \( \xi(\epsilon) \) is given by
\[ \xi(\epsilon) = \xi_1 + O(\epsilon). \] (A.12)

Proof sketch:
Following [4] we consider the auxiliary function \( \Phi \) given by
\[ \Phi(\xi, \epsilon) = \frac{1}{\epsilon^2} \cdot \left[ P(\xi_0 + \xi \cdot \epsilon, \epsilon) - P(\xi_0 + \xi \cdot \epsilon, 0) \right]. \] (A.13)
Show that \( \epsilon = 0 \) is a removable singularity by expending \( P(\xi_0 + \xi \cdot \epsilon, \epsilon) \) as Taylor series in the second argument up to second order and \( D_\epsilon P(\xi_0 + \xi \cdot \epsilon, 0) \) in the first argument. Then apply the implicit function theorem at \( \Phi(\xi_1, 0) = 0 \) to gain the existence of a unique function \( \xi(\epsilon) \) that has the required properties.

Finally, we inspect the stability of the fixed points:

**Lemma 3.** Let there be a three times continuously differentiable map \( P(\xi, \epsilon) : \Lambda \rightarrow \Sigma_0 \) such that
\[ P(\xi, 0) \equiv \xi, \quad \text{for all } \xi \in \Sigma_0. \] (*)
Let there be \( \xi_0 \in \Sigma_0 \) and \( 0 < \epsilon_{\text{max}} \leq \epsilon_{\text{def}} \) and a continuously differentiable function \( \xi(\epsilon) : [0, \epsilon_{\text{max}}) \rightarrow \mathbb{R}^n \) such that
\[ P(\xi_0 + \xi(\epsilon) \cdot \epsilon, \epsilon) \equiv \xi_0 + \xi(\epsilon) \cdot \epsilon, \quad \text{for all } 0 \leq \epsilon < \epsilon_{\text{max}}. \] (A.14)
Assume every eigenvalue \( \lambda_i \) of \( D_\epsilon D_\xi P(\xi_0, 0) \) has a negative real part.
Then, there exists \( 0 < \epsilon_{\text{stab}} < \epsilon_{\text{max}} \) such that for every \( \epsilon \in (0, \epsilon_{\text{stab}}) \) the closed trajectory based at \( F(\epsilon) = \xi_0 + \xi(\epsilon) \cdot \epsilon \) in the corresponding \( \epsilon \)-perturbed system is asymptotically stable.
Proof sketch:
To analyse the stability of the solutions, the eigenvalues \( \mu_i \) of \( D_\xi P(\xi_0 + \xi(\epsilon) \cdot \epsilon, \epsilon) \) are examined to show that for small \( \epsilon > 0 \) eigenvalues are inside the unit cycle.
Due to \( (*) \), in the case \( \epsilon = 0 \) it holds \( P(\Delta \xi + \xi_0, 0) = \Delta \xi + \xi_0 \) or
\[
D_\xi P(\xi_0, 0) = I, \quad \text{(A.15)}
\]
every eigenvalue is equal to 1 and every trajectory is obviously Lyapunov stable, but not asymptotically stable.
Continue by analysing the case of \( \epsilon > 0 \). Expand \( D_\xi P(\xi_0 + \xi(\epsilon) \cdot \epsilon, \epsilon) \) as a Taylor series in \( \epsilon \) around 0:
\[
D_\xi P(\xi_0 + \xi(\epsilon) \cdot \epsilon, \epsilon) = D_\xi P(\xi_0, 0) + \epsilon \cdot \left( D_\epsilon D_\xi P(\xi_0, 0) + D_\xi(\xi_0) \cdot D_\epsilon(0) + O(\epsilon) \right), \quad \text{(A.16)}
\]
where \( O(\epsilon) \in O(\epsilon) \). Equation (A.15) leads to \( D_\xi D_\xi P(\xi_0, 0) = 0 \) and therefore
\[
D_\xi P(\xi_0 + \xi(\epsilon) \cdot \epsilon, \epsilon) = I + \epsilon \cdot \left( D_\epsilon D_\xi P(\xi_0, 0) + O(\epsilon) \right). \quad \text{(A.17)}
\]
Due to the assumptions, every eigenvalue \( \lambda_i \) of \( D_\epsilon D_\xi P(\xi_0, 0) \) has a negative real part. Define
\[
r_{\min} := \min_{i=1,...,n} \| \text{Re}(\lambda_i) \| > 0. \quad \text{(A.18)}
\]
Due to the continuity of eigenvalues with respect to perturbations of the matrix, there exists \( \epsilon_{r_{\min}/2} > 0 \) such that for all \( 0 \leq \epsilon < \epsilon_{r_{\min}/2} \) every eigenvalue \( \tilde{\lambda}_i(\epsilon) \) of the matrix \( D_\epsilon D_\xi P(\xi_0, 0) + O(\epsilon) \) is included in
\[
D(r_{\min}/2) := \bigcup_{i=1}^{n} D(\lambda_i, r_{\min}/2) \subset \{ z \in \mathbb{C} \mid \text{Re}(z) < 0 \}. \quad \text{(A.19)}
\]
The eigenvalues \( \mu_i \) of \( D_\xi P(\xi_0 + \xi(\epsilon) \cdot \epsilon, \epsilon) \) are connected to the eigenvalues \( \tilde{\lambda}_i \) of \( D_\epsilon D_\xi P(\xi_0, 0) + O(\epsilon) \) by
\[
\mu_i(\epsilon) = 1 + \epsilon \cdot \tilde{\lambda}_i(\epsilon). \quad \text{(A.20)}
\]
The function
\[
\gamma: \{ z \in \mathbb{C} \mid \text{Re}(z) < 0 \} \rightarrow (0, \infty), \quad \gamma(z) := \frac{2 \cdot \text{Re}(z)}{\text{Re}(z)^2 + \text{Im}(z)^2}, \quad \text{(A.21)}
\]
is continuous and it holds
\[ (1 + \epsilon \cdot z) \in D(0, 1), \quad \text{for all } 0 < \epsilon < \gamma(z). \quad (A.22) \]
On the compact set \( D(r_{\text{min}}/2) \gamma(z) \) has a minimum \( \epsilon_{\gamma_{\text{min}}} \). For
\[ 0 < \epsilon < \epsilon_{\text{stab}} := \min \{ \epsilon_{\gamma_{\text{min}}} , r_{\text{min}}/2 \}, \quad (A.23) \]
it holds that
\[ \tilde{\lambda}_i(\epsilon) \in D(r_{\text{min}}/2), \quad \text{for all } 0 < \epsilon < \epsilon_{\text{stab}}, \quad (A.24) \]
and
\[ \mu_i(\epsilon) = (1 + \epsilon \cdot \tilde{\lambda}_i(\epsilon)) \in D(0, 1), \quad \text{for all } 0 < \epsilon < \epsilon_{\text{stab}}. \quad (A.25) \]
Thus, trajectories are asymptotically stable.

Appendix A.3. Concretisation for ABS

In the preceding section, we have presented general results assuming the special property \((*)\) of the Poincaré map. There are various types of systems that exhibit this property. In this section, we examine one type of dynamical systems as resulting from our ABS design. We extend the results to specific conclusions for this system which can be applied to the ABS model. Assume a phase space given by
\[ \Omega := \mathbb{R}^n \times \mathcal{V}, \]
\[ \mathcal{V} := [V_0, V_1], \quad (A.26) \]
where \( V_0 < V_1. \)

We introduce two switching surfaces
\[ \Sigma_0 = \{(x, v) \in \Omega \mid v = V_0\}, \]
\[ \Sigma_1 = \{(x, v) \in \Omega \mid v = V_1\}. \quad (A.27) \]

Let the dynamics depend on a state variable \( u \in \{1, 2\} \) and either be given by
\[ \dot{x} = \epsilon \cdot f_1(x, v, \epsilon), \]
\[ \dot{v} = k_1 + \epsilon \cdot g_1(x, v, \epsilon), \quad (A.28) \]

\[ ^{14}\text{The system is a special case of more general, cylindrical phase spaces such as } \mathcal{V} := \{V_0 + [(v - V_0) \mod V_2] \mid v \in \mathbb{R}\}, \text{ where } V_0 < V_1 < V_2 \text{ and } \Sigma_0 = \{(x, v) \in \Omega \mid v = V_0 \equiv V_2\} \text{ and } \Sigma_1 = \{(x, v) \in \Omega \mid v = V_1\} \text{ that can also be analysed by this method.} \]
for \( u = 1 \) or

\[
\begin{align*}
\dot{x} &= \epsilon \cdot f_2(x, v, \epsilon), \\
\dot{v} &= k_2 + \epsilon \cdot g_2(x, v, \epsilon),
\end{align*}
\]  
(A.29)

for \( u = 2 \) where \( k_1 > 0, \ k_2 < 0 \). The trajectory moves up and down between the two switching surfaces: Whenever the switching surface \( \Sigma_1 \) is hit, the dynamics shall switch to \( u = 2 \), at \( \Sigma_0 \) to \( u = 1 \). We directly find that property (*) is fulfilled. We treat \( \Sigma_0 \) as a Poincaré section and introduce the Poincaré map

\[
P : \Lambda \subset \Sigma_0 \times [0, \epsilon_{def}) \to \Sigma_0
\]  
(A.30)

where \( \Lambda \) consists of those \( (x, \epsilon) \in \Sigma_0 \times [0, \epsilon_{def}) \) that have a first return. On these conditions, the Poincaré map of a point \( x \in \Sigma_0 \) is given by

\[
P(x, \epsilon) = x(T_c, x, V_0, \epsilon),
\]  
(A.31)

where \( T_c = T_1 + T_2 \) is the time the trajectory needs to reach \( \Sigma_1 \) and return to \( \Sigma_0 \). To split up the flow into the two flows corresponding to the two states of the system we introduce the functions \( x_1 \) and \( x_2 \) that describe the flow in each of the states. Thus, we find

\[
x(T_c, x, V_0, \epsilon) = x_2(T_2(x_1(T_1(x, \epsilon), x, V_0, \epsilon), \epsilon), x_1(T_1(x, \epsilon), x, V_0, \epsilon, V_1, \epsilon), \]  
(A.32)

and can resort to the conditions given in lemma 1, 2 and 3. The derivatives of \( P \) result in partial derivatives of the flow functions.

Most of the derivatives cross out and we find simple and computable expressions of the conditions:

**Proposition 3.** With \( V_2 := V_0 \) it holds that

\[
D_\epsilon P(\xi_0, 0) = \frac{1}{k_1} \cdot \int_{V_0}^{V_1} f_1(\xi_0, \tau, 0)d\tau + \frac{1}{k_2} \cdot \int_{V_1}^{V_2} f_2(\xi_0, \tau, 0)d\tau,
\]  
(A.33)

and

\[
D_\xi D_\epsilon P(\xi_0, 0) = \frac{1}{k_1} \cdot \int_{V_0}^{V_1} D_x f_1(\xi_0, \tau, 0)d\tau + \frac{1}{k_2} \cdot \int_{V_1}^{V_2} D_x f_2(\xi_0, \tau, 0)d\tau.
\]  
(A.34)

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As far as a computable description for the first order approximation $\xi_1$ is concerned, we find

$$
\xi_1 = -\left[ D_x D_t x_1(\xi_1) + D_x D_t x_2(\xi_2) \right]^{-1} \cdot \left[ D_t D_x x_1(\xi_1) \cdot D_t T_1(\xi_0, \epsilon) + \frac{1}{2} \cdot D_t D_x x_1(\xi_1) \cdot D_t T_2(\xi_0, \epsilon) + \frac{1}{2} \cdot D_t D_x x_2(\xi_2) + D_t D_x x_2(\xi_2) \cdot D_t x_1(\xi_1) \right].
$$

(A.35)

where

$$
D_t D_x x_1(\xi_1) = f_i(\xi_0, V_i, 0), \quad \text{for } i = 1, 2,
$$

(A.36)

$$
D_t D_x x_1(\xi_1) = \frac{2}{k_i^2} \cdot \int_{V_i-1}^{V_i} \left[ k_i \cdot D_t f_i(\xi_0, \tau, 0) + D_x f_i(\xi_0, \tau, 0) \cdot \int_{V_i-1}^{\tau} f_i(\xi_0, s, 0) ds + D_v f_i(\xi_0, \tau, 0) \cdot \int_{V_i-1}^{\tau} g_i(\xi_0, s, 0) ds \right] d\tau,
$$

(A.37)

for $i = 1, 2$,

$$
D_t T_i(\xi_0, 0) = -\frac{1}{k_i^2} \cdot \int_{V_i-1}^{V_i} g_i(\xi_0, \tau, 0) d\tau, \quad \text{for } i = 1, 2,
$$

(A.38)

and

$$
D_t x_1(\xi_1) = \frac{1}{k_1} \cdot \int_{V_0}^{V_1} f_1(\xi_0, \tau, 0) d\tau,
$$

(A.39)

$$
D_t D_x x_2(\xi_2) = \frac{1}{k_2} \cdot \int_{V_1}^{V_2} D_x f_2(\xi_0, \tau, 0) d\tau,
$$

(A.40)

$$
\zeta_i = \left( \frac{V_i - V_0}{k_i}, \xi_0, V_{i-1}, 0 \right), \quad \text{for } i = 1, 2.
$$

We refer to Köppen (2013) for further details.

**Remark 4.** With the identification $\lambda = x$, $T_b = v$ and $V_0 = T_{b_{\min}}$ and $V_1 = T_{b_{\max}}$ the ABS design exhibits the structure introduced (A.26) and we can directly apply the propositions in our analysis.

**Remark 5.** In the case $g_i \equiv 0$ formula (A.35) greatly simplifies.
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