Research Plan

<u>OBJECTIVES</u>. Spiking neurons are at the heart of many computational models of the brain. A neuron generates an action potential in the response to an activity of other neurons to which the neural cell is coupled. A resonate-and-fire neuron model is one of the most fundamental conceptual models that can generate bursts of spikes as opposed to single spikes. As bursts are needed to increase the reliability of communication between neurons (sending a short burst of spikes instead of a single spike increases the chances that at least one of the spikes avoids synaptic transmission failure), the study of bursting oscillations is of importance in neuroscience. This project investigates the occurrence and transitions of spiking and bursting oscillations in resonate-and-fire models.

The resonate-and-fire neuron model is a planar impulsive system of differential equations which describes the interplay between the neural cell membrane potential v and the recovery current u. There are two types of impulses in the model: 1) (Fig. 2d) *time-dependent impulses*, which represent the pulses of current coming from other neurons at certain instances of time; 2) (Fig. 1b) *state-dependent impulses*, which reset any trajectory that reaches the threshold $v = v_{th}$ to a certain phase point (same point for all trajectories). Each such reset generates a current (spike), which further excites subsequent neurons. Oscillations which do not reach the threshold are called subthreshold oscillations. Oscillations which interact with the threshold are called spiking oscillations. The occurrence of subthreshold oscillations is well understood through the Hopf bifurcation theory. Analytic theory of bifurcations of spiking oscillations is a largely open problem.

The goal of the project is to develop a theory capable to control both **1)** transitions of subthreshold oscillations to spiking oscillations and **2)** transitions of spiking oscillations to bursting oscillations, in various types of resonate-and-fire models. The project will develop analytical formulas that neurologists can use in order to analyze and influence the spiking response of neurons to periodic pulse excitation.

To achieve the goal the PI will use a fundamental observation (see e.g. Coombes et al [1]) that new spikes in periodically spiking neurons can appear and disappear either from an equilibrium or through a singular situation where the respective oscillating solution is tangent to the threshold $v = v_{th}$ at some time of the period. Such a situation is known as *grazing incident* in the dynamical systems literature (Makarenkov-Lamb [7]). The project will expose the available knowledge (including his own results) in the context of integrate-and-fire model.

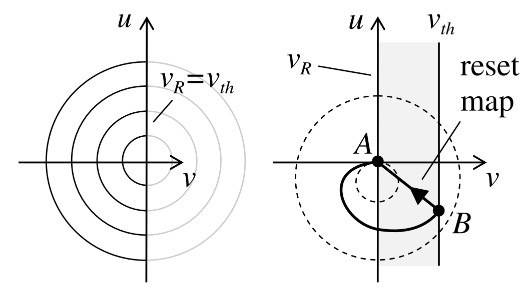
<u>METHODS</u>. The unified model for resonate-and-fire and integrate-and-fire neurons reads as ([2, 4])

$$\begin{array}{lll} \dot{v} = f(v) - u + I, & \text{reset:} \ \ v(t+0) = v_R, \ u(t+0)) = u_R(u(t-0)), \ \text{if } v(t) = v_{th}, \\ \dot{u} = av - bu, & \text{pulse excitation:} \ \ v(t+0) = v(t-0) + \delta, \ \text{if } t = t_* + kT, \ k \in \mathbb{N}, \end{array} \tag{1}$$

where I is a constant current, T and t_* are periods and phase of pulse input. Under different sets of assumptions grazing bifurcations of spikes in the response of (1) to periodic stimulus studied in [1, 3, 8]. However, none of these works addressed explicit (i.e. analytically variable) conditions on the coefficients of the right-hand-side (1) which ensure one or another spiking behavior. As a matter of fact, conditions for generic grazing bifurcations always contain some explicit quantities, which can not be expressed in terms of the right-hand-sides. That is why the current project introduces and investigates completely new (non-generic) scenarios for the occurrence of spiking oscillations, which lead to analytically verifiable conditions. Designing non-generic grazing bifurcations turned out to be very useful e.g. in atomic force microscopy.

In what follows we propose different scenarios of introducing the small parameter ε, for which bifurcation of spiking oscillations occurs when ε crosses 0.

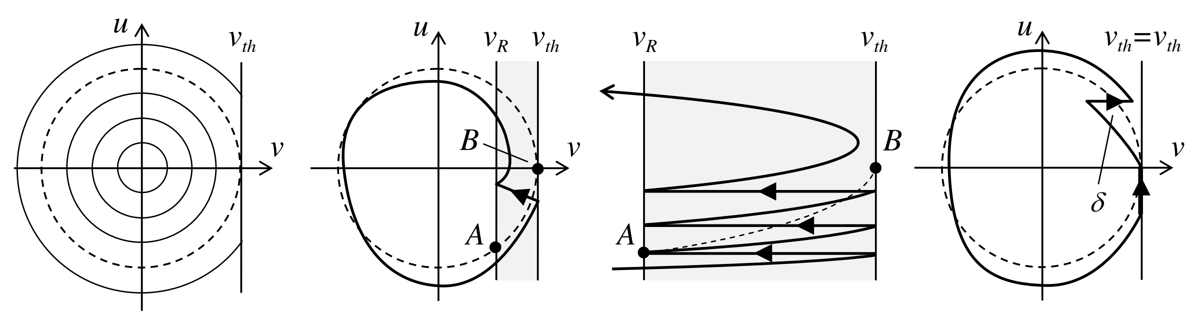
Scenario 1: *Occurrence of spiking oscillations from an equilibrium* (nonsmooth Hopf bifurcation). The PI anticipates that scenario 1 takes place for autonomous systems (1) (i.e. $\delta = 0$) with $v_{th} = \varepsilon\tilde{v}_{th}$, $v_R = 0$, $u_R(u) = u - \varepsilon\tilde{d}$, $I = \varepsilon\tilde{I}$. In other words, we initially consider $v_{th} = 0$, i.e. the equilibrium of (1) with $\varepsilon = 0$ belongs the threshold $v = v_{th}$ (non-genericity). A spiking limit cycle (with an option for bursting, if $\tilde{d} \neq 0$) bifurcates from the origin as the distance between the lines $v = 0$ and $v = v_{th}$ increases, see Fig. 1. We expect that the greater number of spikes per burst will be achieved by diminishing the constant $\tilde{d} > 0$. There will be used a border splitting bifurcation theory that the PI just developed ([6]) to study bifurcations of cycles constrained by the boundaries that shrink together as $\varepsilon \to 0$. In particular, [6] features an expansion (in powers of ε) of the point transformation map that maps A to B along the arc $A \mapsto B$ of Fig. 1b.



(a) ε=0: semicircles (b) $\varepsilon > 0$: spiking

Figure 1: Simplest Scenario 1. The origin transforms to spiking solution (bold curve) when ε increases through zero. Bursting can be design as in Fig. 2c.

Scenario 2: *Occurrence of spiking oscillations from grazing cycle of a center* (nonsmooth Melnikov bifurcation). It is proposed to consider $f(v) = \varepsilon\hat{f}(v)$, $b = \varepsilon\tilde{b}$, $a = \omega_0 + \varepsilon\tilde{\omega}$, $T = 2\pi/\omega_0$, $v_R = v_{th} - \varepsilon\tilde{\Delta}$, $u_R(u) = u - d$, $b = \varepsilon\tilde{b}$, $I = 0$, $\delta = \varepsilon\tilde{\delta}$. As in Scenario 1, we view the distance between the lines $v = v_t h$ and $v = r_R$ as a small parameter, however this strip is far from the origin that produces spiking oscillations (bursting, if $d > 0$) that surround the origin, see Fig. 2. The spiking solution approaches the unit circle of radius v_{th} centered at the origin as $\varepsilon \to 0$. This limiting cycle $(v_0(t), u_0(t))$ is a member of a center, that (1) possesses when $\varepsilon = 0$. In particular, we will impose a condition (*non-generic condition*) which ensures that the only cycle of the center which produces resonances (as ε increases through 0) is the cycle $(v_0(t), u_0(t))$. We anticipate that the T-periodic excitation will lock the period, i.e. $2\pi/(\omega_0 + \varepsilon\tilde{\omega}) \neq T$. A result of this type with $\tilde{\Delta} = 0$ was investigated in PI's paper [5] (occurrence of stick-slip oscillations in a dry friction oscillator). The spiking oscillations in Scenario 2 will be obtained by unfolding the dynamics of [5] for $\tilde{\Delta} > 0$.



(a) ε= 0: cycles (center) (b) $\varepsilon > 0$: spiking (c) $\varepsilon > 0$: bursting (d) $\varepsilon > 0$: spiking

Figure 2: Scenario 2 for different $\tilde{f}$, $\tilde{b}$, $\tilde{\omega}$, $\tilde{\delta}$, $\tilde{\Delta}$. Arrows "▶" stay for instantaneous jumps. The dash cycle transforms to spiking solution (bold curve) when ε increases through zero.

Scenario 3: *Resonance-induced spiking oscillations.* Let us now have $f(v) = \varepsilon\tilde{f}(v)$, $b = \varepsilon\tilde{b}$, $a = \omega_0 + \varepsilon\tilde{\omega}$, $I \neq 0$, $T = 2\pi/\omega_0$, $v_R = 0$, $\delta = \varepsilon\tilde{\delta}$. Consider an auxiliary system by removing the threshold $v = v_{th}$ from (1). If we don't impose the non-generic condition of Scenario 2 then a resonance T-periodic solution $(v_\varepsilon(t), u_\varepsilon(t))$ of the auxiliary system can occur from any cycle $(v_0(t), u_0(t))$ of the center depending on $\tilde{f}$, $\tilde{b}$, $\tilde{\omega}$, $\tilde{\delta}$. First, we will derive the conditions to ensure that the cycle $(v_0(t), u_0(t))$ overlaps $v = v_{th}$. Second, we will show that such a setting implies the existence of spiking oscillations in the full system, as Fig. 3 illustrates. The PI will use the resonance theory (i.e. the method of Van der Pol) for impulsive differential equations [9].

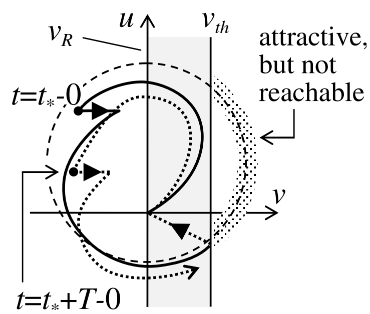


Figure 3: Resonance spiking solution (begins as solid curve, continues as dotted) under Scenario 3 over two rounds about the origin.

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