Ergodic Theory. Homework 3.

(due $\sim$ 02.11.2015)

1. We will say that node a is n-connected to b if there exists a path of length n, connecting a and b. We will denote this by $a \xrightarrow{n} b$. Period of an element a for the irreducible matrix P is equal to $r_a = \gcd\{n : a \xrightarrow{n} a\}$. Prove that r_a does not depend on a.

2. Suppose that P is irreducible but not aperiodic (period r is greater than 1). Then fix some vertex a_0 and denote by S_k the set of all vertices which are $k \mod r$-connected to a_0. Check that S_k are pairwise disjoint. Prove that P^r maps S_k to itself for any k. Prove that P^r restrickted to S_k is irreducible and aperiodic. Use ergodic theorem for irreducible aperiodic matrices to prove that in the periodic case one gets $\lim\limits_{n\to\infty} \frac{1}{n} \sum\limits_{k=1}^{n} P_{ij}^n = \vec{p}_j$.

3. Let P be an irreducible matrix and Q is conjugated to P. Is it true that Q is also irreducible? (Hint Is it true that product of two irreducible matrices is also irreducible?) Same question for aperiodicity.

Some experiments.

4. Prove that Arnold's cat transformation is measure preserving. Check that four-vertex graph, constructed in the previous hometask does not provide the Markov decomposition for this transformation (i.e. there exists inadmissible sequences of the four admissible steps)

   Check that the following construction provides the Markov decomposition for the Arnold's cat transform.

   Let ξ_s and ξ_u denote the stable and unstable directions for the matrix $T = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Lines corresponding to these directions split the basic unit square onto five parts constituting two rectangles with the sides parallel to ξ_s and ξ_u. Check that transformation T acts on these parts as a Markov shift.

5. Construct the Markov decomposition consisting of two squares for the transformation $T_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.