Ergodic Theory. Homework 4.

(due $\sim$ 02.25.2015)

1. Prove that the product of irrational rotations $R_{\alpha_1}, \dots R_{\alpha_n}$ is ergodic if and only if $\alpha_1, \dots, \alpha_n$ are indeendent over rational numbers.

2. Let $\Delta_1, \dots, \Delta_n, \dots$ are the intervals constructed by the induced transformations from the rigid rotation by angle α having continued fraction $[0; a_1, a_2, a_3, \dots]$. Prove that $\dfrac{|\Delta_n|}{|\Delta_{n+1}|} = a_n$.

3. Let $\alpha = [a_0; a_1, a_2, \dots]$. Denote $\dfrac{p_n}{q_n} = [a_0; a_1, \dots a_n]$.

   - Prove that
   $$\begin{bmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{bmatrix} = \begin{bmatrix} a_0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_n & 1 \\ 1 & 0 \end{bmatrix}$$
   - Prove that q_n and p_n satisfy the following properties: 1) $q_n \geqslant (\sqrt{2})^{n-2}$, $p_n \geqslant (\sqrt{2})^{n-2}$, 2) $p_n q_{n-1} - p_{n-1} q_n = (-1)^{n+1}$, 3) $|\alpha - \frac{p_n}{q_n}| < \frac{1}{q_n q_{n+1}}$.

Some experiments.

4.
   - Construct a continued fraction for the number $\sqrt{2}$. (Hint $\sqrt{2} = 1 + (\sqrt{2} - 1)$ and $(a-b)(a+b) = a^2 - b^2$).
   - Construct a continued fraction for the number $\dfrac{\sqrt{5}-1}{2}$.
   - Prove that if for some integre numbers p and q one gets $\alpha^2 + p\alpha + q = 0$ then continued fraction for α is periodic.
   - Prove that if continued fraction for α is periodic then α is a quadratic irrational (i.e is a solution of the equation $a\alpha^2 + b\alpha + c = 0$ for some integer a, b and c)

5. First terms in the continueds fraction for π is $[3; 7]$ which provides $\pi \simeq \dfrac{22}{7}$. Next term is 15 which gives approximant $\dfrac{355}{113}$. Find two next terms. How does the approximation error behave? What are the continued fraction for e?