Cyclic Evasion in the Three Bug Problem

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Abstract. In this note, we present a simple proof that three bugs involved in cyclic evasion converge to an equilateral triangle configuration. The approach relies on an energy-type estimate that makes use of a new inequality for the triangle.

The problem of the cyclic pursuit or $n-$bug problem is a classical one; see *e.g.*, an article by Klamkin and Newman, "Cyclic pursuit or the three bugs problem" [**5**]. Because of applications in robotics and availability of more powerful computers, there has recently been some revival of interest in the n-bug problem and various generalizations [**3, 4, 6, 7, 9**].

The related problem of cyclic evasion, where each bug runs with the unit velocity directly away from one other bug, corresponds to reversing the time. This problem has also received some attention, especially in the computer science literature; see *e.g.*, [**2**]. It was observed numerically and verified with heuristic arguments, that asymptotically, the n bug configuration converges either to a regular (convex or star) polygon or to a line configuration. In particular, it is widely believed that in the case of three bugs, the limiting configuration is that of an equilateral triangle; however, no proof is available in the literature (to our knowledge). The goal of this expository note is to give a complete and short proof in this simplest case of three bugs[1]. An apparently new inequality for a triangle is proved and used to establish the convergence.

To fix the notation, let each bug be represented by $\mathbf{r_i}(t) = (x_i(t), y_i(t)) \in \mathbb{R}^2$, $i = A, B, C$, where t is the time parameter. The bugs' velocities are then given by

$$\frac{d\mathbf{r_i}}{dt} = \frac{\mathbf{r_i} - \mathbf{r_{i+1}}}{|\mathbf{r_i} - \mathbf{r_{i+1}}|},$$

where $i + 1$ is understood as cyclic shift $A \to B \to C \to A$.

Using elementary geometry (see the Figure 1), one obtains the following equations; see *e.g.*, [**1, 5**] for the angles and side lengths:

$$\dot{a} = 1 + \cos\beta, \quad \dot{b} = 1 + \cos\gamma, \quad \dot{c} = 1 + \cos\alpha, \tag{1}$$

$$\dot{\alpha} = \frac{\sin\gamma}{b} - \frac{\sin\alpha}{c}, \quad \dot{\beta} = \frac{\sin\alpha}{c} - \frac{\sin\beta}{a}, \quad \dot{\gamma} = \frac{\sin\beta}{a} - \frac{\sin\gamma}{a}. \tag{2}$$

If we denote the perimeter by

$$P = a + b + c,$$

then

$$V(a, b, c) = \dot{P} = 3 + \cos\alpha + \cos\beta + \cos\gamma \geqslant 0,$$

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[1]The case $n = 2$ is completely trivial as two points always belong to a line.

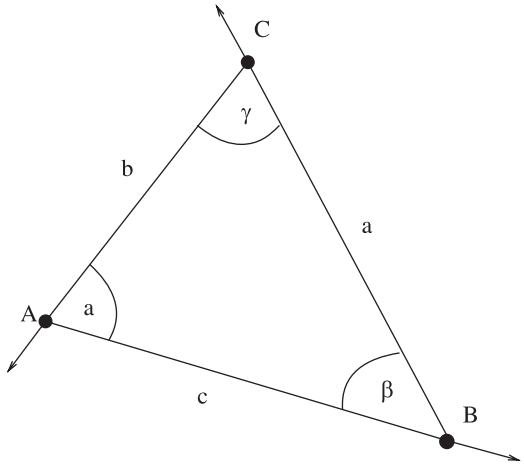

Figure 1. The three bug configuration

and

$$\dot{V} = \frac{\sin^2\alpha}{c} + \frac{\sin^2\beta}{a} + \frac{\sin^2\gamma}{b} - \frac{\sin\alpha\sin\beta}{c} - \frac{\sin\beta\sin\gamma}{a} - \frac{\sin\gamma\sin\alpha}{b}.$$

Using the law of sines, we replace angles by lengths, and clearing denominators we obtain:

$$\dot{V} = \frac{S^2}{4a^3b^3c^3}\left(a^3b + b^3c + c^3a - a^2b^2 - b^2c^2 - c^2a^2\right),$$

where S is the area of the triangle. It turns out that the fourth order homogeneous polynomial in the brackets is non-negative, provided the three variables are the side lengths in a triangle. Thus, we will have $\dot{V} \geqslant 0$.

Proposition. *Let $a > 0$, $b > 0$ and $c > 0$ be sides of a triangle, then*

$$W(a, b, c) = a^3b + b^3c + c^3a - (a^2b^2 + b^2c^2 + c^2a^2) \geqslant 0 \tag{3}$$

and $W(a, b, c) = 0$ if and only if $a = b = c$.

Proof. Since a, b and c are sides of a triangle, we have $a + b \geqslant c$, $b + c \geqslant a$ and $c + a \geqslant b$. Introducing change of variables $a = x + y$, $b = y + z$, $c = z + x$, using that $x, y, z \geqslant 0$, and substituting in the equation, we obtain

$$W = x^3y + y^3z + z^3x - xyz(x + y + z).$$

Next, we use the weighted inequality for arithmetic and geometric mean [**8**]:

$$\frac{w_1F_1 + w_2F_2 + w_3F_3}{w_1 + w_2 + w_3} \geqslant \sqrt[w]{F_1^{w_1}F_2^{w_2}F_3^{w_3}}, \tag{4}$$

where $w = w_1 + w_2 + w_3$ and equality occurs only if $F_1 = F_2 = F_3$, for positive weights. We first prove that

$$\frac{w_1 x^3 y + w_2 y^3 z + w_3 z^3 x}{w_1 + w_2 + w_3} \geqslant x^2 yz$$

using the appropriate weights:

$$\begin{cases} 3w_1 + w_3 = 2w_1 + 2w_2 + 2w_3 \\ w_1 + 3w_2 = w_1 + w_2 + w_3 \\ w_2 + 3w_3 = w_1 + w_2 + w_3. \end{cases}$$

Hence, for $w_1 = 4$, $w_2 = 1$, $w_3 = 2$, we get

$$\frac{4}{7}x^3 y + \frac{1}{7}y^3 z + \frac{2}{7}z^3 x \geqslant x^2 yz.$$

Permuting the variables twice, we obtain two more inequalities. Adding all three inequalities, we obtain the desired one. The equality occurs only if $x^3 y = y^3 z = z^3 x$, which is equivalent to $x = y = z$, provided $xyz \neq 0$. Note that if at least one variable vanishes, say $x = 0$, and equality occurs, then $y^3 z = 0$ and either $a = x + y = 0$ or $c = z + x = 0$, contradicting our assumption. ■

Clearly, $\dot{V} = 0$ for the line configuration because $S = 0$, and $\dot{V} = 0$ for the equilateral triangle because $a = b = c$ and W vanishes. Now, we prove convergence to the equilateral triangle.

Theorem 1. *Suppose the initial configuration of the bugs $A(0), B(0), C(0)$ is not a line configuration, then as $t \to \infty$, $A(t), B(t), C(t)$ approach equilateral configuration.*

Proof. If the initial configuration is an equilateral triangle $a = b = c$, $\alpha = \beta = \gamma$, then we are done. Thus, we assume the triangle is nondegenerate and not equilateral. The function V is defined in the set

$$D = \{\alpha > 0, \beta > 0, \gamma > 0, \alpha + \beta + \gamma = \pi\},$$

which is itself a triangle.

In the interior of D, the function V has a single extremum $\alpha = \beta = \gamma = \pi/3$ as can be verified by the method of Lagrange multipliers, which leads to the relation

$$\sin\alpha = \sin\beta = \sin\gamma \Rightarrow \alpha = \beta = \gamma = \pi/3.$$

We used the constraint $\alpha + \beta + \gamma = \pi$ and that the angles do not vanish. It is easy to check that $V(\pi/3, \pi/3, \pi/3) = 9/2$ is the only maximum of V in D.

We claim that an orbit starting in the interior of D, will stay away from the boundary of D. Substituting $\gamma = \pi - \alpha - \beta$, we have

$$V(\alpha, \beta) = 3 + \cos\alpha + \cos\beta - \cos(\alpha + \beta).$$

Because of the threefold symmetry of the domain D, we only need to consider one segment of the boundary of D, *e.g.*, $\gamma = 0$. At $\gamma = 0$, we have

$$\frac{\partial V}{\partial \alpha}(\gamma = 0) = -\sin\alpha, \quad \frac{\partial V}{\partial \beta}(\gamma = 0) = -\sin\beta, \tag{5}$$

which implies that V increases in any direction inside D from any point on the boundary segment $\gamma = 0$ provided $\alpha, \beta \neq 0, \pi$. Near the corners of D, such degeneracy actually takes place. Consider then a neighborhood of the corner $\alpha = 0, \beta = 0$. Using Taylor expansions, we have

$$V(\alpha, \beta) = 4 + \alpha\beta + o(\alpha^2 + \beta^2),$$

which together with (5) proves that V increases in any direction inside D from any point on the boundary. Since $\dot{V} > 0$ in the interior of D away from $\alpha = \beta = \gamma = \pi/3$, no orbit can approach the boundary.

Thus, $V(t)$ increases and $V(t) \to V_0 \leq V_{\max}$. Then we must also have $\dot{V}(t) \to 0$, and there exists a sequence of times $t_n \to \infty$ such that $(\alpha(t_n), \beta(t_n), \gamma(t_n)) \to (\pi/3, \pi/3, \pi/3)$. Then $V(t) \to V_{\max}$ and the orbit $(\alpha(t), \beta(t), \gamma(t))$ must converge to $(\pi/3, \pi/3, \pi/3)$. ■

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