FINAL EXAM - SOLUTIONS

Math 1314-501 College Algebra

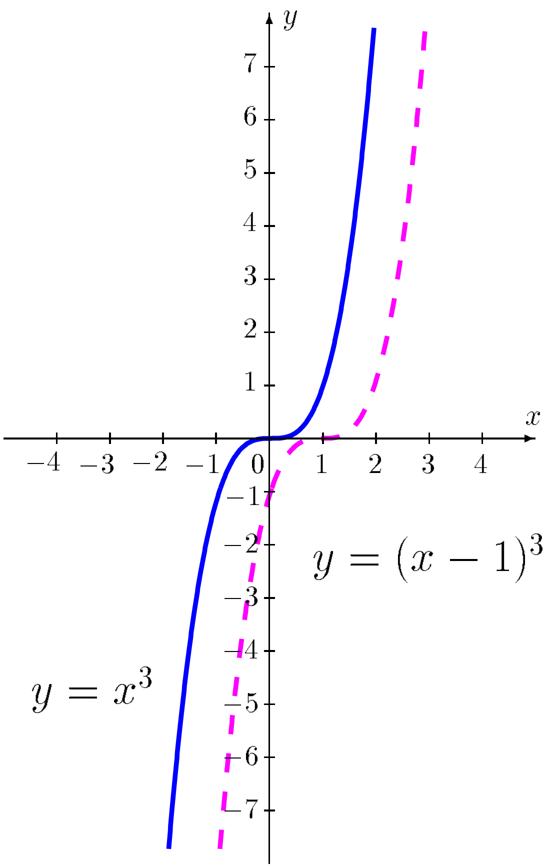
1. Here is the graph of the function $y = x^3$. Use the basic graphing principles to sketch the graph of the function $f(x) = (|x| - 1)^3$.

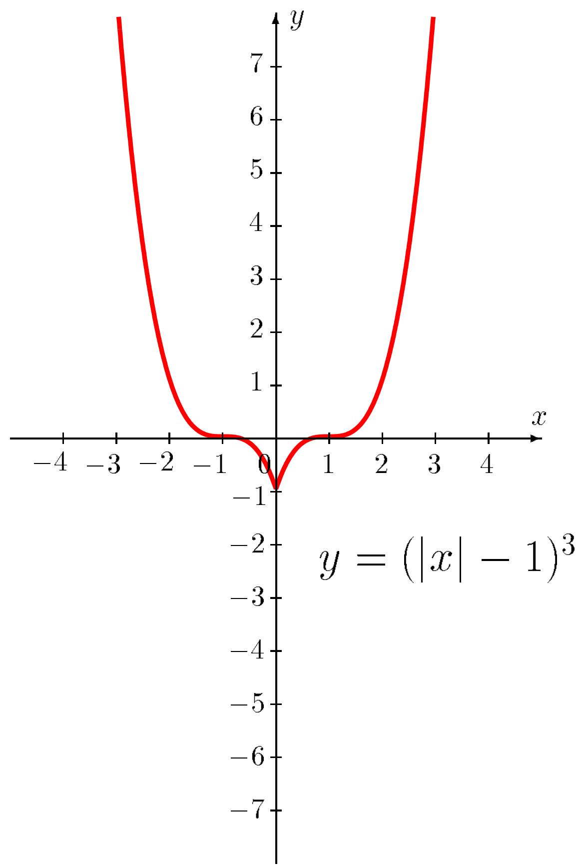
   SOLUTION:

   (1) Shift the graph to the right by 1 unit;

   (2) Reflect the right part (where $x \geq 0$) about the y-axis.

   Notice that the function is even because $f(-x) = f(x)$. Therefore, its graph should be symmetric about the y-axis.





2. Consider the quadratic function $g(x) = 2x^2 + 7x + 6$. Compute coordinates of the vertex, x-intercepts, and y-intercepts, if they exist.

SOLUTION:

Here $A = 2$, $B = 7$, and $C = 6$.

(1) Vertex:

$$\begin{aligned} x &= -\frac{B}{2A} = -\frac{7}{4} \\ y &= C - \frac{B^2}{4A} = 6 - \frac{49}{8} = \frac{48}{8} - \frac{49}{8} = -\frac{1}{8} \\ \text{or} \\ y &= 2\left(-\frac{7}{4}\right)^2 + 7\left(-\frac{7}{4}\right) + 6 = \frac{49}{8} - \frac{49}{4} + 6 = 6 - \frac{49}{8} = \frac{48}{8} - \frac{49}{8} = -\frac{1}{8} \end{aligned}$$

(2) x-intercept:

$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} = \frac{-7 \pm \sqrt{49 - 48}}{4} = \frac{-7 \pm 1}{4} = -2 \text{ and } -1.5$$

(3) y-intercept: $C = 6$

ANSWER: vertex $\left(-\frac{7}{4}, -\frac{1}{8}\right)$, x-intercepts $(-2, 0)$ and $(-1.5, 0)$, y-intercept $(0, 6)$.

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3. Consider a function

$$f(x) = \frac{3x + 7}{x - 5}$$

Find equations for the horizontal and vertical asymptotes.

SOLUTION:

(1) The vertical asymptote, where the function does not exist, $x = 5$.

(2) To find the horizontal asymptote, apply polynomial division:

$$\begin{array}{r|l} & \quad 3 \\ \hline x - 5 & 3x + 7 \\ & 3x - 15 \\ \hline & \quad\;\; 22 \end{array}$$

So, $f(x) = 3 + \frac{22}{x - 5}$. The horizontal asymptote is $y = 3$.

ANSWER: vertical asymptote $x = 5$, horizontal asymptote $y = 3$.

4. Consider again a function

$$f(x) = \frac{3x+7}{x-5}$$

Find the inverse function and verify that $f(f^{-1}(y)) = y$.

<u>SOLUTION</u>:

Solve the equation $y = \dfrac{3x+7}{x-5}$ *for* x.

$$\begin{aligned} y(x-5) &= 3x+7 \\ xy - 5y &= 3x+7 \\ xy - 3x &= 5y+7 \\ x(y-3) &= 5y+7 \\ x &= f^{-1}(y) = \frac{5y+7}{y-3} \end{aligned}$$

Verify:

$$f(f^{-1}(x)) = \frac{3\left(\frac{5y+7}{y-3}\right)+7}{\left(\frac{5y+7}{y-3}\right)-5} = \frac{3(5y+7)+7(y-3)}{5y+7-5(y-3)} = \frac{15y+21+7y-21}{5y+7-5y+15} = \frac{22y}{22} = y$$

<u>ANSWER</u>: the inverse function is $f^{-1}(y) = \dfrac{5y+7}{y-3}$.

5. Draw the graph of a function:

$$h(x) = -\frac{1}{2}x^2 + 2x + 6$$

SOLUTION:

Complete a square,

$$h(x) = -\frac{1}{2}x^2+2x+6 = -\frac{1}{2}\left\{x^2 - 4x - 12\right\} = -\frac{1}{2}\left\{(x^2 - 4x + 4) - 16\right\} = -\frac{1}{2}\left\{(x-2)^2 - 16\right\}$$

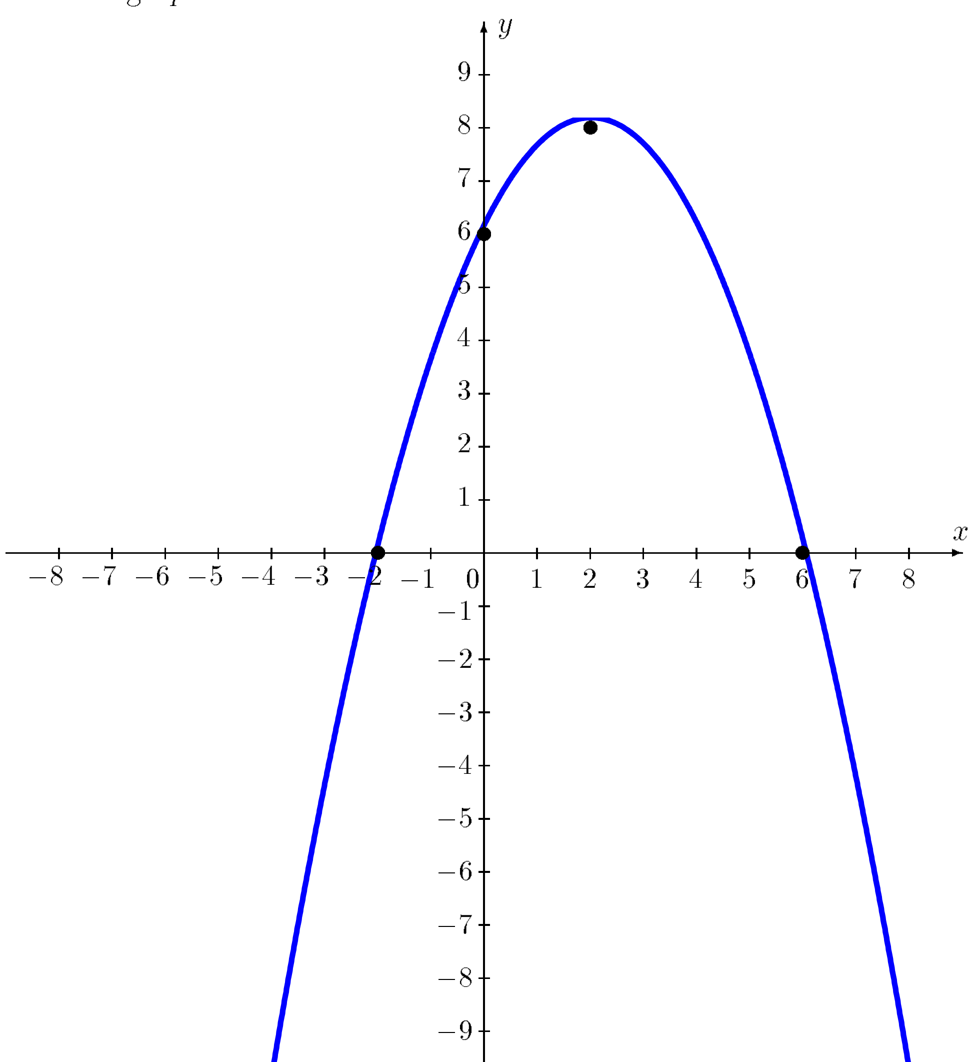
So, we have a "cap"-shape parabola (concave because $A = -\frac{1}{2} < 0$), squeezed towards x-axis (because $|A| < 1$), with the vertex at $x = 2$, $y = h(2) = 8$.

For better accuracy, we can also compute the intercepts,

y-intercept at $h(0) = C = 6$;

x-intercepts at $\dfrac{-2 \pm \sqrt{(-2)^2 - 4(-\frac{1}{2})(6)}}{2(-\frac{1}{2})} = \dfrac{-2 \pm \sqrt{4+12}}{-1} = 2 \pm 4 = -2$ *and* 6.

We can now draw the graph.



6. Compute without a calculator:

   (a) $\log_2\left(\frac{1}{4}\right)$

   (b) $\log_{27} 3$

   <u>SOLUTION</u>:

   (a) $\log_2\left(\frac{1}{4}\right) = -2$ *because* $2^{-2} = \frac{1}{4}$

   (b) $\log_{27} 3 = \frac{1}{3}$ *because* $27^{1/3} = \sqrt[3]{27} = 3.$

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7. Population of town X increases according to a formula

$$P(t) = 50,000 \cdot 2^{\,t/10}$$

where t is the number of years since the year 2000.

During what year will the population of X reach 200,000 people?

<u>SOLUTION</u>:

After t years since 2000, we have

$$P(t) = 50,000 \cdot 2^{\,t/10} = 200,000.$$

Solve it for t,

$$\begin{aligned} 2^{\,t/10} &= 4 \\ 2^{\,t/10} &= 2^2 \\ t/10 &= 2 \\ t &= 20 \end{aligned}$$

<u>ANSWER</u>: after 20 years, i.e., during year 2020, the population of X reaches 200,000 people.

8. John has 200 feet of fence that he will use to surround a rectangular playground for his children. Let x be one of the sides.

   (a) Write the playground area as a function of x.

   (b) Compute the maximum possible area that John can surround with his fence.

   SOLUTION:

   (a) Let x be one of the sides of the playground. The other side is $(100 - x)$ so that the total perimeter is

$$P = 2x + 2(100 - x) = 200 \textit{ feet of fence}$$

   Then, the playground area is

$$f(x) = x(100 - x) = -x^2 + 100x.$$

   (b) The maximum of this parabola equals

$$y = C - \frac{B^2}{4A} = 0 - \frac{100^2}{4(-1)} = 25,000 \textit{ sq. feet}$$

   ANSWER: the area is $f(x) = -x^2 + 100x$; the maximum area is 25,000 sq. feet.