Markov Chains

Markov chain = discrete-time, discrete-state Markov stochastic process

It is described by

(1) initial state distribution

$$\boldsymbol{P}\{X(0) = x\} = P_0(x)$$

(2) transition probabilities

$$P\{X(n+1) = j \mid X(n) = i\} = P_n(i \to j)$$

If $P_n(i \to j) = p_{ij}$ is independent of n, it is a *stationary* Markov chain

k-step transition probabilities

$$p_{ij}(k) = P\left\{X(n+k) = j \mid X(n) = i\right\}$$

Based on $\begin{pmatrix} P_0(1) \\ P_0(2) \\ \cdots \end{pmatrix}$ *and* $\begin{pmatrix} p_{11} & p_{12} & p_{13} & \cdots \\ p_{21} & p_{22} & p_{23} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}$,

how to compute

$$p_{ij}(k)$$

$$\boldsymbol{P}\left\{X_n = x\right\}$$

$$\lim_{n\to\infty} \boldsymbol{P}\left\{X_n = x\right\} \qquad ?$$

MATRICES

$$A = \left\{A_{ij},\ i = 1, ..., r, j = 1, ..., c\right\}$$

$$= \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1c} \\ A_{21} & A_{22} & \cdots & A_{2c} \\ \cdots & \cdots & \cdots & \cdots \\ A_{r1} & A_{r2} & \cdots & A_{rc} \end{pmatrix}$$

is a matrix with r rows and c columns

i = row number
j = column number

Multiplying a row by a column

$A = (A_1, \ldots, A_n) =$ "1 by n" matrix

$$B = \begin{pmatrix} B_1 \\ \vdots \\ B_n \end{pmatrix} = \text{"}n \text{ by 1" matrix}$$

Then

$$AB = (A_1, \ldots, A_n) \begin{pmatrix} B_1 \\ \vdots \\ B_n \end{pmatrix} = \sum_{i=1}^{n} A_i B_i$$

Example: 3 hrs 25 min 45 sec = 12345 sec because

$$(3\ 25\ 45) \begin{pmatrix} 3600 \\ 60 \\ 1 \end{pmatrix} = 12345$$

A product of matrices

If $A =$ "k by m" matrix
$B =$ "m by n" matrix

then $C = AB =$ "k by n" matrix where

$$\begin{aligned} C_{ij} &= \sum_{h=1}^{n} A_{ih} B_{hj} \\ &= \begin{pmatrix} i^{th} \ row \\ of \ A \end{pmatrix} \begin{pmatrix} j^{th} \ column \\ of \ B \end{pmatrix} \end{aligned}$$

Example

$$\begin{pmatrix} 2 & 6 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 9 & -3 \\ -3 & 1 \end{pmatrix} = \text{interesting answer}$$

Markov chains: transition probability *matrix*

$$P = \left\{p_{ij}\right\} = \left\{P(i \to j),\ \begin{matrix} i = 1, \ldots, n \\ j = 1, \ldots, n \end{matrix}\right\}$$

$$= \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{pmatrix}$$

k-step transition probability *matrix*

$$\begin{aligned} P(k) &= \left\{p_{ij}(k)\right\} \\ &= \left\{P(i \to j \text{ in } k \text{ steps }),\ \begin{matrix} i = 1, \ldots, n \\ j = 1, \ldots, n \end{matrix}\right\} \end{aligned}$$

Two-step transition probability matrix

$$\begin{aligned} p_{ij}(2) &= \boldsymbol{P}\{X(2) = j \mid X(0) = i\} \\ &= \sum_k P(i \to k) P(k \to j) \\ &\qquad \text{(Law of Total Probability)} \\ &= (p_{i1}, \ldots, p_{in}) \begin{pmatrix} p_{1j} \\ \vdots \\ p_{nj} \end{pmatrix} \end{aligned}$$

Hence,

$$P(2) = P \cdot P = P^2$$

k-step transition probability matrix

$$P(k) = \underbrace{P \cdot P \cdot \ldots \cdot P}_{k \text{ times}} = P^k$$

which means

$$p_{ij}(k) = \boldsymbol{P}\{X(k) = j \mid X(0) = i\}$$

$$= \sum_s \sum_t \sum_u \cdots \sum_z P(i \to s) P(s \to t) \cdots P(z \to j)$$

Distribution of $X(k)$

If
$$
\begin{aligned}
P_0 &= (P_0(1), \cdots, P_0(n)) = \text{pmf of } X(0), \\
P_k &= (P_k(1), \cdots, P_k(n)) = \text{pmf of } X(k), \\
P &= \left\{p_{ij}\right\} = \text{transition probability matrix},
\end{aligned}
$$

then by the Law of Total Probability,

$$
\begin{aligned}
P_k(j) &= \boldsymbol{P}\{X(k) = j\} \\
&= \sum_i \boldsymbol{P}\{X(0) = i\} \boldsymbol{P}\{X(k) = j \mid X(0) = i\} \\
&= \sum_i P_0(i) p_{ij}(k)
\end{aligned}
$$

Hence

$$P_k = P_0 P^k$$

Steady-state (limiting) probabilities

After very many transitions,
what is the distribution of $X(k)$?

That is, $\lim_{k\to\infty} P_0 P^k = \ ?$

A Markov chain is regular if

$$p_{ij}(k) > 0$$

for some k and all i, j

Fact. For a regular Markov chain, there is a limit

$$\lim_{k\to\infty} P^k = \Pi,$$

with all $\Pi_{ij} > 0$ and all rows of Π being equal.

Steady-state distribution

Thus,

$$\Pi = \begin{pmatrix} \pi_1 & \pi_2 & \dots & \pi_n \\ \pi_1 & \pi_2 & \dots & \pi_n \\ \vdots & \vdots & \vdots & \vdots \\ \pi_1 & \pi_2 & \dots & \pi_n \end{pmatrix}$$

$\pi = (\pi_1, \pi_2, \dots, \pi_n)$ = steady-state distribution

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Then $\lim_k \boldsymbol{P}\{X(k) = x\}$ is found as

$$P_k(x) = (P_0(1), \dots, P_0(n)) \begin{pmatrix} \pi_x \\ \pi_x \\ \vdots \\ \pi_x \end{pmatrix} = \pi_x$$

and it does not depend on the initial state!

Computing π

$\Pi = \lim_k P^k$, therefore, $\Pi P = \Pi$

Computation

$\pi = (\pi_1, \ldots, \pi_n)$ solves the system

$$\begin{cases} \pi P & = & \pi \\ \sum_i \pi_i & = & 1 \end{cases}$$

($\pi =$ eigenvector of P)