Equally Likely Outcomes

$$S = \text{ sample space } = \{O_1, O_2, ..., O_n\}$$

$\{O_k\}$ are **equally likely** if

$$P(O_1) = P(O_2) = ... = P(O_n) = \frac{1}{n}$$

Then, for any event E,

$$P(E) = \frac{\text{number of } O_k \in E}{n}$$

$$= \frac{\text{number of } \textbf{favorable} \text{ outcomes}}{\textbf{total} \text{ number of outcomes}}$$

Equally likely	Not equally likely
random sampling results of a coin toss a card is drawn gambling, roulette	errors in different modules results of a football game grades for a course market ups and downs

Conditional Probability

= probability of A if B is known to occur

Notation: $P\{A \mid B\}$

How to compute it?

In the case of *equally likely outcomes*, count only $O_k \in B$.

Then, instead of

$$P(A) = \frac{\text{\# of } O_k \in A}{\text{\# of } O_k \in S},$$

compute

$$P\{A \mid B\} = \frac{\text{\# of } O_k \in A \cap B}{\text{\# of } O_k \in B} = \frac{P(A \cap B)}{P(B)}$$

$$P\{A \mid B\} = \frac{P(A \cap B)}{P(B)}$$

therefore, for the general case,

$$P(A \cap B) = P\{A \mid B\}\, P(B) = P\{B \mid A\}\, P(A)$$

Independence

Events A and B are **independent** if

$$P\{A \mid B\} = P(A)$$

i.e., the fact that B occurs does not affect the probability of A.

Then

$$P(A) = \frac{P(A \cap B)}{P(B)} \quad \Rightarrow \quad P(A \cap B) = P(A)P(B)$$

Bayes Formula

We have

$$P\{A \mid B\}\, P(B) = P(A \cap B) = P\{B \mid A\}\, P(A)$$

$$\Downarrow$$

$$P\{A \mid B\} = \frac{P\{B \mid A\}\, P(A)}{P(B)}$$

Formula of Total Probability

We have

$$A = \{A \cap B\} \cup \{A \cap \bar{B}\}$$

Therefore

$$\begin{aligned} P(A) &= P(A \cap B) + P(A \cap \bar{B}) \\ &= P\{A \mid B\}\, P(B) + P\{A \mid \bar{B}\}\, P(\bar{B}) \end{aligned}$$

General case. Let $B_1, ..., B_n$ be mutually exclusive and exhaustive (partition of S). Then

$$P(A) = \sum_{j=1}^{n} P(A \mid B_j) P(B_j)$$

Use it when $P(A \mid B_j)$ are easier to compute than $P(A)$.

The Bayes Formula:

$$P\{A \mid B\} = \frac{P\{B \mid A\}\, P(A)}{P\{B \mid A\}\, P(A) + P\{B \mid \bar{A}\}\, P(\bar{A})}$$