CONTINUOUS DISTRIBUTIONS

Continuous variables may assume any value in an interval (a, b), $(a, +\infty)$, $(-\infty, +\infty)$, *etc.*

Examples: installation time
waiting time
service time
temperature
weight
percentage stock return
... ...

Recall: discrete variables assume only a finite or countable number of isolated values

Therefore, $P\{X = x\} = 0$ for any continuous variable X.

Only

$$F(x) = P\{X \leq x\} \text{ (cdf) },$$

$$F(b) - F(a) = P\{a < X \leq b\}$$

are not trivial.

Corollary: $P\{X \leq x\} = P\{X < x\}$

Probability Density

Let
$F(x) = P\{X \leq x\}$, nondecreasing, *continuous*.

Define

$$f(x) = F'(x) = \frac{dF}{dx}$$

$F(x)$ is a cumulative distribution function (cdf)

$f(x)$ is a probability density function (pdf)

Computing probabilities

So, $f(x) = F'(x)$.

Hence, by Fundamental Theorem of Calculus,

$$\begin{aligned}\int_{x=a}^{x=b} f(x)dx &= F(b) - F(a) \\ &= P\{X \leq b\} - P\{X \leq a\} \\ &= P\{a < X < b\}\end{aligned}$$

$$\int_a^b f(x)dx = F(b) - F(a)$$

Special cases:

$$\begin{aligned}\int_{-\infty}^{b} f(x)dx &= F(b) = P\{X \leq b\} \\ \int_{a}^{\infty} f(x)dx &= 1 - F(a) = P\{X > a\} \\ \int_{-\infty}^{\infty} f(x)dx &= P\{-\infty < X < \infty\} = 1\end{aligned}$$

So, $\int_{-\infty}^{\infty} f(x)dx = 1$ always

Probabilities are **areas** under the density curve

Distributions

Discrete	Continuous
$p(x) = P(X = x)$ (pmf)	$f(x) = F'(x)$ (density)
$P(X \in A) = \sum_{x \in A} p(x)$	$P(X \in A) = \int_A f(x)dx$
$\sum_x p(x) = 1$	$\int_{-\infty}^{\infty} f(x)dx = 1$

$$F(x) = P(X \leq x)$$ (cdf)

Joint and marginal distributions

$p(x) = \sum_y p(x,y)$	$f(x) = \int f(x,y)dy$
$p(y) = \sum_x p(x,y)$	$f(y) = \int f(x,y)dx$

Independence

$p(x,y) = p(x)p(y)$	$f(x,y) = f(x)f(y)$

Remarks

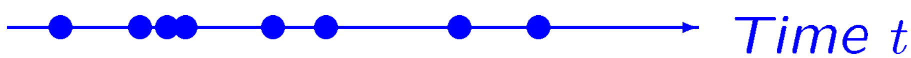
Cdf $F(x)$ - continuous, non-decreasing from 0 to 1

Pdf $f(x)$ - non-negative, area = 1

$f(x)$ can be obtained as a limit of frequency histograms

EXPONENTIAL DISTRIBUTION

Rare events...



N_1 = # of events in 1 min = Poisson(λ)
N_2 = # of events in 2 min = Poisson(2λ)
....
N_t = # of events in t min = Poisson($t\lambda$)

X = Time between events = Exponential(λ)
X_1 = Time of the first event = Exponential(λ)

Exponential distribution is continuous.

<u>Examples</u>:	waiting time	installation time
	lifetime	time between calls
	time between rare natural disasters	

Exponential distribution function and density

$$
\begin{aligned}
1 - F(x) = P\{X > x\} \\
&= P\{\text{ more than } x \text{ minutes until next event }\} \\
&= P\{\text{ no events within } x \text{ minutes }\} \\
&= P\{N_x = 0\} \qquad \text{where } N_x \text{ is Poisson}(\lambda x) \\
&= e^{-\lambda x}\frac{(\lambda x)^0}{0!} \\
&= e^{-\lambda x}
\end{aligned}
$$

So, Exponential cdf: $F(x) = 1 - e^{-\lambda x}$

Exponential pdf: $f(x) = F'(x) = \lambda e^{-\lambda x}$

$$(x > 0)$$

Memoryless property
of Exponential distribution

What is the chance that component A lives x hours?

Let X = *time to failure* = *Exponential*(λ)

$$P(X > x) = 1 - F(x) = e^{-\lambda x}$$

Component B did not fail for t hours. What is the chance it lives the next x hours?

$$P\{X > t + x \mid X > t\} = \frac{P\{X > t + x \ \cap X \ \ > t\}}{\{X > t\}}$$

$$= \frac{P\{X > t + x\}}{\{X > t\}} = \frac{e^{-\lambda(t+x)}}{e^{-\lambda t}} = e^{-\lambda x}$$

Loss of memory: $X > t$ does not matter!

Facts about distributions:

$$\left.\begin{aligned} &Continuous \\ &Memoryless \end{aligned}\right\} \Rightarrow Exponential$$

$$\left.\begin{aligned} &Discrete \\ &Memoryless \end{aligned}\right\} \Rightarrow Geometric$$