STOCHASTIC PROCESSES

Let $t \in T$ = time

$\omega \in S$ = outcome,
element of the sample space

$X(t, \omega)$ = **stochastic process**

Discrete T ⇒ discrete time process
Connected T ⇒ continuous time process

$X \in \mathcal{X}$ = states of the process

Discrete $\mathcal{X}$ ⇒ discrete state process, chain
Connected $\mathcal{X}$ ⇒ continuous state process

For any t $\Rightarrow$ $X_t(\omega) =$ random variable
For any ω $\Rightarrow$ $X_\omega(t) =$ function of t
(path, trajectory, realization)

Examples

Temperature

Stock value

Number of jobs in a queue

Number of internet connections

Football score

Poisson process

Binomial process

Brownian motion

Counting processes

They *count* events.

Therefore, $X \in \mathcal{X} = \{0, 1, 2, 3, ...\}$;
$X(t)$ is non-decreasing.

Examples:

Binomial process

Poisson process

Markov processes

$X(t)$ is a **Markov process** if
for any $t_1 < \ldots < t_n < t$,

$$\boldsymbol{P}\{X(t) \in A \mid X(t_1) = x_1, \ldots, X(t_n) = x_n\}$$

$$= \boldsymbol{P}\{X(t) \in A \mid X(t_n) = x_n\}$$

That is,

$$\boldsymbol{P}\{\text{ future } \mid \text{ past, present }\}$$

$$= \boldsymbol{P}\{\text{ future } \mid \text{ present }\}$$

Markov dependence:
"Future depends on the past only through the present"

Bernoulli process

Discrete time $t = 1, 2, 3, ...$

Let $X(1), X(2), ... =$ independent Bernoulli(p) variables

Then $X(t) =$ **Bernoulli process**

It is discrete-time, discrete-state, Markov, but not counting.

It is *stationary*, i.e., the distribution of $X(t)$ does not change,

$\boldsymbol{E}X(t) = p$, $\text{Var}X(t) = p(1-p)$.

Binomial process

Let $X(t)$ = *Bernoulli process*

Binomial process = partial sums

$$S(t) = X(1) + ... + X(t).$$

For each t, $S(t)$ is Binomial(t,p),

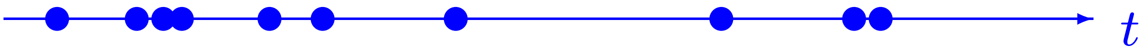
$\boldsymbol{E}S(t) = tp$, Var$S(t) = tp(1-p)$.

It is discrete-time, discrete-state, Markov, counting, but not stationary.

It counts events that occur in discrete time (frames).

Interarrival times = times between successive events are $Geometric(p)$

Poisson process



$X(t) =$ continuous-time process counting *"rare events"*, with properties:

- $\boldsymbol{P}\{X(t+h) - X(t) = 1\}$
  $= \boldsymbol{P}\{$ 1 event in [t,t+h] $\}$
  $= \lambda h + o(h)$, as $h \to 0$

- $\boldsymbol{P}\{X(t+h) - X(t) > 1\}$
  $= \boldsymbol{P}\{$ more than 1 event in [t,t+h] $\}$
  $= o(h)$, as $h \to 0$

- For $t_1 < t_2 < t_3 < t_4$, the *increments* $X(t_2) - X(t_1)$ and $X(t_4) - X(t_3)$ are **independent**

Poisson process is continuous-time, discrete-state, Markov.

It is stationary if λ is constant.
where $\lambda = \boldsymbol{E}(\#\ events/min)$

Binomial process	$\longrightarrow$	**Poisson process**
(discrete time; frame = Δ)	$\Delta \to 0$ $p \to 0$ $\lambda = const$	(continuous time)

Distributions

Let $X(t)$ = Binomial process
= number of events during time t
= number of events during $n = t/\Delta$ frames

$$X(t) = Binomial\left(n = \frac{t}{\Delta}, p\right) \to Poisson(\lambda t)$$

as $\Delta \to 0$,
hence $p \to 0$, $n = t/\Delta \to \infty$, $np = tp/\Delta = \lambda t$

$$\boldsymbol{E}X(t) = \lambda t, \qquad \mathrm{Var}X(t) = \lambda t$$

Interarrival times

$$\text{Interarrival time} = \min\{t, X(t) \geq 1\}$$

Counting process	Interarrival times
Binomial Poisson	$\Delta\cdot$Geometric(p) Exponential(λ)

Let $T_1, T_2, \ldots =$ successive interarrival times

$$\begin{array}{ccc} \boldsymbol{P}\{T_1 + \ldots + T_n > t\} & = & \boldsymbol{P}\{X(t) < n\} \\ \uparrow & & \uparrow \\ Gamma(n,\lambda) & & Poisson(\lambda t) \\ \downarrow & & \downarrow \\ \boldsymbol{P}\{T_1 + \ldots + T_n \leq t\} & = & \boldsymbol{P}\{X(t) \geq n\} \end{array}$$

(Gamma-Poisson formula)