OPRE 6201 : 1. Formulation

1 Mathematical Models

Model: A structure which has been built purposefully to exhibit features and characteristics of some other object such as a DNA model in biology, a building model in civil engineering, a play in a theatre and a mathematical model in OR.

Why to build models?

1. Improved understanding and communication
2. Experimentation
3. Standardization for analysis

Example 1: $F = m \cdot a$ is a physical model studying the relationship between force (F), mass (m) and accelaration (a). Note that the model does not capture the friction. We say "friction" is abstracted out in this model for "computability".

We use variables and equations to construct mathematical models. Thus, Example 1 illustrates a mathematical model.

Common features of mathematical models:

- Abstraction, are details overlooked?
- Computability, can the model be manipulated with ease?
- Inputs, data requirements
- Uncertainty, are inputs and relationships between them uncertain?
- Decision horizons, flexibility, risk considerations ...

In deterministic mathematical models, there is no uncertainty. Then, the important concerns are abstraction and computability.

How much to trust?

- Too little. Claim: some data (especially utilities and social values) can not be quantified
- Too little. Claim: uncertain inputs
- Too much. Models are not divine, question the assumptions and the output. regard the output as an option.

2 Model Components

2.1 The Decision Variables

Decision variables capture the level of activities that the model studies. Decision makers have some freedom (subject to Constraints, see below) to assign numerical values to decision variables. For example number of bolts (screws) a machining plant produces in a week, denoted by B (S), is a common decision variable. Letting, say, $B = 5000$ and $S = 7200$, we specify that 5000 bolts and 7200 screws are produced in a week. Solving a mathematical model means finding these numerical values for decision variables to minimize or maximize an *objective function* in the presence of *constraints*.

2.2 The Objective Function

With mathematical models, we wish to maximize or minimize a quantity such as cost, profit, risk, net present value, number of employees, customer satisfaction, etc. The quantity we wish to maximize or minimize is known as *objective function*. Deciding on the correct objective in practical situations is not trivial. At one extreme there may be no clear objectives, at the other there may be multiple objectives. There are no recipes for the process of coming up with the correct objective however, one develops a feeling for this process by studying various examples.

Suppose that the machining plant makes a profit of \$13 from every 1000 bolts and \$15 from every 1000 screws. Then the objective function of maximizing weekly profit is:

$$\text{Objective Function :} \quad \max 0.013B + 0.015S.$$

What are the units of this objective function? Every time you write an objective function or a constraint, check the units.

2.3 The Constraints

Constraints represent the limitations such as available capacity, daily working hours, raw material availability, etc. Sometimes constraints are also used to represent relationships between decision variables.

Suppose that the machining plant has 120 (with 3 lathes) hours of turning capacity per week. It has 36 hours of grinding capacity per week. Also two people work halftime and one person works fulltime for bolt and screw production making available number of manpower capacity 80 hours per week. Table below lays out turning, grinding and manpower hours needed to produce 1000 bolts and 1000 screws.

Activity	# of hrs required per 1000 bolts	# of hrs required per 1000 screws
Turning	3	4
Grinding	2	1
Manpower	1	3

If we produce B many bolts and S many screws, we need $3B/1000$ turning hours for bolts and $4S/1000$ turning hours for screws. Total turning hours needed is $3B/1000 + 4S/1000$, which has to be less than 120 hours available for turning per week:

$$\text{Turning Constraint :} \quad 3B/1000 + 4S/1000 \leq 120.$$

Similarly we can write down two other constraints one for grinding capacity and one for manpower capacity:

$$\text{Grinding Constraint :} \quad 2B/1000 + S/1000 \leq 36.$$

$$\text{Manpower Constraint}: \; B/1000 + 3S/1000 \leq 80.$$

Since both the numbers of bolts and screws are nonnegative numbers, we also add:

$$\text{Nonnegativity Constraints}: \; B \geq 0 \; , \; S \geq 0.$$

We obtain a formulation for the machining plant by putting the objective function and four constraints together. As an afterthought, do you think we really need the Nonnegativity Constraints? (Hint: Would you think removing Nonnegativity Constraints changes the solution?) We will revisit this question in much more detail.

Another instructive exercise is reformulating the machine plant problem after letting B and S be the number of bolts and screws in thousands. This is known as scaling a model. Scaling can improve the accuracy of solution techniques but this is outside the scope of this course.

2.4 Linear Programming Assumptions

1. Proportionality: Contribution of each activity to an objective function and constraints is **proportional** to the level of that activity. For example, if 1000 bolts require 3 turning hours, 100 bolts require 0.3 hours and 2000 bolts require 6 hours. This assumption fails when there is (dis)economies of scale.
2. Additivity: Individual contribution of different activities can be **summed** up to obtain an objective function and constraints. For example, turning constraint is obtained by summing turning hours required by bolts and screws. This assumption fails when activities are not independent.
3. Certainty: Each parameter in the formulation is known for **sure**.
4. Divisibility: Decision variables can take integer as well as **fractional values**.

Without Proportionality or Divisibility assumptions we have Nonlinear Programs. Without Certainty assumption we have Stochastic Programs. Without Divisibility assumption we have Integer programs.

3 A Production Planning Problem

Suppose a production manager is responsible for scheduling the monthly production levels of a certain product for a planning horizon of twelve months. For planning purposes, the manager was given the following information:

- The total demand for the product in month j is d_j, for $j = 1, 2, \ldots, 12$. These could either be targeted values or be based on forecasts.
- The cost of producing each unit of the product in month j is c_j (dollars), for $j = 1, 2, \ldots, 12$. There is no setup/fixed cost for production.
- The inventory holding cost per unit for month j is h_j (dollars), for $j = 1, 2, \ldots, 12$. These are incurred at the end of each month.
- The production capacity for month j is m_j, for $j = 1, 2, \ldots, 12$.

The manager's task is to generate a production schedule that minimizes the total production and inventory-holding costs over this twelve-month planning horizon.

To facilitate the formulation of a linear program, the manager decides to make the following simplifying assumptions:

1. There is no initial inventory at the beginning of the first month.

2. Units scheduled for production in month j are immediately available for delivery at the beginning of that month. This means in effect that the production rate is infinite.

3. Shortage of the product is not allowed at the end of any month.

To understand things better, let us consider the first month. Suppose, for that month, the planned production level equals 100 units and the demand, d_1, equals 60 units. Then, since the initial inventory is 0 (Assumption 1), the ending inventory level for the first month would be 0+100-60=40 units. Note that all 100 units are immediately available for delivery (Assumption 2); and that given $d_1 = 60$, one must produce no less than 60 units in the first month, to avoid shortage (Assumption 3). Suppose further that $c_1 = 15$ and $h_1 = 3$. Then, the total cost for the first month can be computed as: $15 \cdot 100 + 3 \cdot 40 = 1380$ dollars.

At the start of the second month, there would be 40 units of the product in inventory, and the corresponding ending inventory can be computed similarly, based on the initial inventory, the scheduled production level, and the total demand for that month. The same scheme is then repeated until the end of the entire planning horizon.

3.1 The Decision Variables

The manager's task is to set a production level for each month. Therefore, we have twelve decision variables:

- x_j = the production level for month j, $j = 1, 2, \ldots, 12$.

3.2 The Objective Function

Consider the first month again. From the discussion above, we have:

- The production cost equals $c_1 x_1$.
- The inventory-holding cost equals $h_1(x_1 - d_1)$, provided that the ending inventory level, $x_1 - d_1$, is nonnegative.

Therefore, the total cost for the first month equals $c_1 x_1 + h_1(x_1 - d_1)$.
For the second month, we have:

- The production cost equals $c_2 x_2$.
- The inventory-holding cost equals $h_2(x_1 - d_1 + x_2 - d_2)$, provided that the ending inventory level, $x_1 - d_1 + x_2 - d_2$, is nonnegative. This follows from the fact that the starting inventory level for this month is $x_1 - d_1$, the production level for this month is x_2, and the demand for this month is d_2.

Therefore, the total cost for the second month equals $c_2 x_2 + h_2(x_1 - d_1 + x_2 - d_2)$.

Continuation of this argument yields that:

- The total production cost for the entire planning horizon equals

$$\sum_{j=1}^{12} c_j x_j \equiv c_1 x_1 + c_2 x_2 + \ldots + c_{12} x_{12} \,,$$

  where we have introduced the standard summation notation ("$\equiv$" means by definition).

- The total inventory-holding cost for the entire planning horizon equals

$$\begin{aligned}\sum_{j=1}^{12} h_j\left[\sum_{k=1}^{j}(x_k-d_k)\right] &\equiv h_1\left[\sum_{k=1}^{1}(x_k-d_k)\right]+h_2\left[\sum_{k=1}^{2}(x_k-d_k)\right]+\ldots\\ &\quad +h_{12}\left[\sum_{k=1}^{12}(x_k-d_k)\right]\\ &= h_1\left[x_1-d_1\right]+h_2\left[(x_1-d_1)+(x_2-d_2)\right]+\ldots\\ &\quad +h_{12}\left[(x_1-d_1)+(x_2-d_2)+\ldots+(x_{12}-d_{12})\right].\end{aligned}$$

Since our goal is to minimize the total production and inventory-holding costs, the objective function can now be stated as

$$\text{Min} \quad \sum_{j=1}^{12} c_j x_j+\sum_{j=1}^{12} h_j\left[\sum_{k=1}^{j}(x_k-d_k)\right].$$

3.3 The Constraints

Since the production capacity for month j is m_j, we require

$$x_j \leq m_j$$

for $j = 1, 2, \ldots, 12$; and since shortage is not allowed (Assumption 3), we require

$$\sum_{k=1}^{j}(x_k-d_k) \geq 0$$

for $j = 1, 2, \ldots, 12$. This results in a set of 24 functional constraints. Of course, being production levels, the x_j's should be nonnegative.

3.4 LP Formulation

In summary, we have arrived at the following formulation:

$$\begin{aligned}&\text{Min} && \sum_{j=1}^{12} c_j x_j+\sum_{j=1}^{12} h_j\left[\sum_{k=1}^{j}(x_k-d_k)\right]\\ &\textit{Subject to}: && \\ & && x_j \;\leq\; m_j \qquad \textit{for } \; j=1,2,\ldots,12\\ & && \sum_{k=1}^{j}(x_k-d_k) \;\geq\; 0 \qquad \textit{for } \; j=1,2,\ldots,12\\ & && x_j \geq 0 \quad \textit{for } \; j=1,2,\ldots,12\,.\end{aligned}$$

This is a linear program with 12 decision variables, 24 functional constraints, and 12 nonnegativity constraints. In an actual implementation, we need to replace the c_j's, the h_j's, the d_j's, and the m_j's with explicit numerical values.

3.5 An Alternative Formulation for Production Planning

In the above formulation, the expression for the total inventory-holding cost in the objective function involves a nested sum, which is rather complicated. Notice that for any given j, the inner sum in that expression, $\sum_{k=1}^{j}(x_k - d_k)$, is simply the ending inventory level for month j. This motivates the introduction of an additional set of decision variables to represent the ending inventory levels. Specifically, let

- y_j = the ending inventory level for month j, $j = 1, 2, \ldots, 12$;

then, the objective function can be rewritten in the following simpler-looking form:

$$\text{Min} \quad \sum_{j=1}^{12} c_j x_j + \sum_{j=1}^{12} h_j y_j \ .$$

With these new variables, the no-shortage constraints also simplify to $y_j \geq 0$ for $j = 1, 2, \ldots, 12$. However, we now need to introduce a new set of constraints to "link" the x_j's and the y_j 's together.
Consider the first month again. Denote the initial inventory level as y_0; then, by assumption, we have $y_0 = 0$. Since the production level is x_1 and the demand is d_1 for this month, we have $y_1 = y_0 + x_1 - d_1$. Continuation of this argument shows that for $j = 1, 2, \ldots, 12$,

$$y_j = y_{j-1} + x_j - d_j \, ;$$

and these relations should appear as constraints to ensure that the y_j 's indeed represent ending inventory levels. We have, therefore, arrived at the following new formulation:

$$\begin{array}{lllll}
\text{Min} & \sum_{j=1}^{12} c_j x_j + \sum_{j=1}^{12} h_j y_j & & & \\
\textit{Subject to}: & & & & \\
& x_j & \leq & m_j & for\ j = 1, 2, \ldots, 12 \\
& y_j & = & y_{j-1} + x_j - d_j & for\ j = 1, 2, \ldots, 12 \\
& x_j & \geq & 0 & for\ j = 1, 2, \ldots, 12 \\
& y_j & \geq & 0 & for\ j = 1, 2, \ldots, 12.
\end{array}$$

which is a linear program with 24 decision variables, 24 functional constraints, and 24 nonnegative variables.

Although there are twice as many decision variables in the new formulation, both formulations have the same number of functional constraints. We will show in a later section that the total amount of effort necessary to arrive at an optimal solution to a linear program depends primarily on the number of functional constraints. In general, it is not uncommon to have several equivalent formulations of the same problem.

3.6 Remarks

- If Assumption 1 is relaxed, so that the initial inventory level is not necessarily zero, we can simply set y_0 to whatever given value.

- In our formulation, we assumed that there is no production delay (Assumption 2). This assumption can be easily relaxed. Suppose instead there is a production delay of one month; that is, the scheduled production for month j, x_j, is available only after a delay of one month, i.e., in month $j+1$. Then, in the alternative formulation, we can simply replace the constraint $y_j = y_{j-1}+x_j-d_j$ by $y_j = y_{j-1}+x_{j-1}-d_j$ (with $x_0 \equiv 0$), for $j = 1, 2, \ldots, 12$. Of course, for the first month, the given value of y_0 must be no less than d_1; otherwise, the resulting LP will not have any solution.

- Assumption 3 can also be relaxed. If shortages are allowed, we can simply remove the nonnegativity requirements for the y_j's, and introduce a shortage penalty cost of, say, p_j per unit of shortage at the end of month j. We will discuss in a later section how to handle these "unrestricted" variables.

4 A Blending Problem

A salad dressing supplier to major DFW area restaurants is considering using LP for its blending problem. This dressing (market value of \$400/ton) is manufactured by refining raw oils and blending them together. Five types of oils come in two categories. Olive oils: Olive 1 and Olive 2. Corn oils: Corn 1, Corn 2 and Corn 3. Oil prices (\$ per ton) for the coming three months are given as follows:

	Olive 1	Olive 2	Corn 1	Corn 2	Corn 3
Oct	280	390	110	180	130
Nov	290	400	90	190	130
Dec	310	430	100	200	130

During refining process olive and corn oils can not be mixed. The dressing supplier dedicates a separate refining unit to each of the olive and corn oils. In any month, olive (corn) oil refining unit can process at most 190 (270) tons of oil. There are five tanks to store each type of oil separately, each tank has 300 tons of capacity. It is unhealthy to store refined oil. Storage costs per ton per month are \$10 for all types of oil.

The hardness of salad dressings is regulated and has to be within 3 and 6. Generally hardness blends linearly and the hardness of the raw oils are given below:

Olive 1	Olive 2	Corn 1	Corn 2	Corn 3
3.1	2.4	7.2	5.8	6.1

We will formulate this blending problem to maximize supplier's profit.

4.1 Decision Variables

We are interested in quantities of raw oil bought, used and stored in each month. Let $O1B_i$ be the olive oil 1 bought in month i ($i = 1$ corresponds to Oct and so on), similarly define $O1U_i$ and $O1S_i$ as the olive oil 1 used in month i and stored at the end of month i. $O2B_i$, $O2U_i$ and $O2S_i$ refer to bought, used and stored olive oil 2. $C1B_i$, $C1U_i$ and $C1S_i$ refer to bought, used and stored corn oil 1 and so on. Also let D_i be the dressing produced and sold in month i.

Warning: Note that the first step in developing a formulation is clearly identifying decision variables. Without such an identification it is impossible to understand the notation in the objective function and the constraints.

4.2 Objective Function

We want to maximize the (profit = revenue - cost). Revenues are obtained by selling the dressing: $\sum_{i=1}^{3} 400 \cdot D_i$. Cost has two components: raw oil purchase costs and storage costs. Raw oil purchase cost: $280O1B_1 + 390O2B_1 + 110C1B_1 + 180C2B_1 + 130C3B_1 + 290O1B_2 + 400O2B_2 + 90C1B_2 + 190C2B_2 + 130C3B_2 + 310O1B_3 + 430O2B_3 + 100C1B_3 + 200C2B_3 + 130C3B_3$. Storage costs: $10(O1S_1 + O2S_1 + C1S_1 + C2S_1 + C3S_1 + O1S_2 + O2S_2 + C1S_2 + C2S_2 + C3S_2 + O1S_3 + O2S_3 + C1S_3 + C2S_3 + C3S_3)$.

4.3 Constraints

Storage transition constraints: $OjS_i = OjS_{i-1} + OjB_i - OjU_i$ for $j = 1, 2$ and $i = 1, 2, 3$. $CjS_i = CjS_{i-1} + CjB_i - CjU_i$ for $j = 1, 2, 3$ and $i = 1, 2, 3$. For simplicity assume that initially there are no oils on stock, i.e. OjS_0 and CjS_0 are both 0.

Storage tank capacity constraints: For olive oil storage tanks $OjS_i \leq 300$ for $i = 1, 2, 3$ and $j = 1, 2$. For corn oil storage tanks $CjS_i \leq 300$ for $i = 1, 2, 3$ and $j = 1, 2, 3$.

Refining capacity constraints: For olive oil refining $O1U_i + O2U_i \leq 190$ for $i = 1, 2, 3$. For corn oil refining $C1U_i + C2U_i + C3U_i \leq 270$ for $i = 1, 2, 3$.

Hardness constraints:

$$3 \leq \frac{3.1O1U_i + 2.4O2U_i + 7.2C1U_i + 5.8C2U_i + 6.1C3U_i}{D_i} \leq 6 \;\; for\ i = 1, 2, 3.$$

Manipulating above inequlity, we obtain the following inequalities:

$$3.1O1U_i + 2.4O2U_i + 7.2C1U_i + 5.8C2U_i + 6.1C3U_I - 6D_i \leq 0 \;\; for\ i = 1, 2, 3.$$

$$3.1O1U_i + 2.4O2U_i + 7.2C1U_i + 5.8C2U_i + 6.1C3U_I - 3D_i \geq 0 \;\; for\ i = 1, 2, 3.$$

Weight conservation constraint:

$$O1U_i + O2U_i + C1U_i + C2U_i + C3U_j - D_i = 0 \;\; for\ i = 1, 2, 3.$$

Nonnegativity constraints: $OjU_i \geq 0$ for $i = 1, 2, 3$ and $j = 1, 2$. $CjU_i \geq 0$ for $i = 1, 2, 3$ and $j = 1, 2, 3$. $D_i \geq 0$ for $i = 1, 2, 3$.

4.4 Remarks

- In the blending problem, we assumed that the customers purchase all the salad dressings produced. Now suppose that customers commit to purchase at least l_i and at most u_i tons of dressing in month i, modify the formulation accordingly. If there were no refining capacity constraints, would l_i be a redundant parameter (i.e. WLOG, could it be set to 0?), why?

- After solving the blending formulation, you realize that all the decision variables corresponding to oil storage at the end of month 3 are 0. Is this a coincidence? How would you justify this to dressing supplier? (Hint: think about the length of decision horizons.)

5 An Investment Problem

Suppose an investor has $100 on Monday. At the start of **every** day of the week (Monday through Friday), the investor has the following investment opportunity available: If he invests x dollars on that day and matches that initial investment with $x/2$ dollars the next day, then he will receive a total return of $2x$ dollars on the third day. Thus, with a total investment of $1.5x$ dollars, the investor receives $2x$ dollars in two days, a gain of $0.5x$ dollars. The investor wishes to determine an investment schedule that maximizes his total cash on Saturday.

To facilitate the formulation of a linear program, the investor decides to make the following simplifying assumptions:

1. If an initial investment is not matched on the subsequent day, the initial investment is lost.
2. Any return that is due on any given day can be reinvested immediately.
3. Cash carried forward from one day to the next does not accrue interest.

4. Borrowing money is not allowed.

To understand things better, let us consider the following "naive" strategy: Begin with an investment of (2/3)100 dollars on Monday, while putting aside (1/3)100 dollars in anticipation of the necessary second installment on the next day (Assumption 1). On Tuesday, the investor executes the second installment and, consequently, he won't have any remaining cash to initiate a new investment. On Wednesday, the investor receives a total return of 2(2/3)100 dollars. This completes an investment cycle, during which the total amount invested in two installments ($100) grew by a factor of 4/3. Suppose further that the investor immediately reinvests the yield he receives on Wednesday (Assumption 2) in the same manner as in the just-completed investment cycle. Then, a similar analysis shows that he will receive a total yield of $(4/3)(4/3)100 = 177.8$ dollars on Friday. At that point, since any new investment that starts on Friday won't mature until Sunday (a day too late), the investor simply carries this second yield into Saturday, resulting in a final cash position of 177.8 dollars.

How good is this naive strategy? While we managed to complete two full investment cycles, it seems uncomforting to watch cash sit idle from Friday to Saturday. This suggests that we might be able to do better. But how? Note that under the divisibility assumption, the set of possible strategies is a continuum, and hence it cannot even be enumerated. Thus, the task is challenging. To find out what is the best strategy, we now turn to an LP formulation of this problem.

5.1 The Decision Variables

Since a new investment cycle can, potentially, be initiated at the start of each day, the investor needs to determine the magnitudes of these first installments. Therefore, let

- x_j = the amount of new investment at the beginning of Day j, $j = 1, 2, 3, 4$,

where Day 1 corresponds to Monday, Day 2 to Tuesday, and so on. There is no need to introduce x_5, why? We also assume that these decisions are to be made immediately **after** receiving yields (if any) from prior investments.

In principle, it seems sufficient (and it is) to work with these four decision variables alone: On Monday, we would invest x_1 dollars and carry a cash saving of $100 - x_1$ forward to Tuesday. On Tuesday, after executing a second installment of $x_1/2$ dollars, we would have $100 - x_1 - x_1/2$ dollars available for an allocation of a new investment and a new cash saving. A little bit of reflection should be convincing that the picture would become rather complex as we move further into future days. In particular, expressing the amount of cash saving at the end of each day as a function of the entire *history* of past decisions quickly becomes a difficult task.

From the previous production-planning example, we learned that it is sometimes desirable to introduce additional sets of decision variables for the purpose of simplifying the formulation task. Now, imagine yourself being at the start of a given day and ask: What actions do I need to take at this point? The answer is:

1. Cough up half of the amount of new investment (if any) that started in the previous day.
2. Initiate a new investment.
3. Carry the remaining cash (if any) forward to the next day.

Observe that these actions cannot be committed unless we know how much saving is being carried forward from the previous day; and that this information depends on the entire prior investment history in a complicated way. To circumvent this difficulty, it therefore seems desirable to define a new set of variables to represent the daily savings. Formally, let

- s_j = the amount of saving carried forward from Day j to Day $j+1$, $j = 1, 2, 3, 4, 5$.

Note that we can think of the initial capital as a saving from Day 0 to Day 1; that is, let $s_0 = 100$. In summary, we have defined a total of 9 decision variables, four x_j's and five s_j's.

5.2 The Objective Function

On Saturday (Day 6), there are two income streams; one is the yield from the investment cycle that started on Thursday and the other is the saving from Friday. The first contribution equals $2x_4$ and the second, s_5. It is important to realize that yields from all earlier investments are "implicitly captured" into these two terms. Thus, our objective is to:

$$\text{Max } 2x_4 + s_5.$$

Note the simplicity of this objective function. It is a consequence of our choice of the decision variables. In general, it is a good practice to conceptualize the formulation of the decision variables and the objective function jointly.

5.3 The Constraints

In the alternative formulation of the production-planning problem, the production levels and the ending inventory levels were linked together via constraints like $y_j = y_{j-1} + x_j - d_j$ to ensure that they logically correspond to their intended interpretations. This can be viewed as "material" balancing. Here again, we need to ensure proper linkage between the daily new investments and the daily savings.

The basic idea is to balance the cash flow at the beginning of each day. For Day 1, we have $s_0 = 100$ dollars available; and this amount is apportioned into two parts: a new investment and a saving. Thus,

$$s_0 = x_1 + s_1.$$

For Day 2, we have s_1 dollars available; and this is apportioned into three parts: $0.5x_1$, x_2, and s_2. Thus,

$$s_1 = 0.5x_1 + x_2 + s_2.$$

At the beginning of Day 3, we receive a yield of $2x_1$, which is immediately available for reinvestment. It is therefore added into s_2; and the total amount is then divided into three parts as in Day 2. This leads to

$$2x_1 + s_2 = 0.5x_2 + x_3 + s_3.$$

Continuation of this argument yields

$$2x_2 + s_3 = 0.5x_3 + x_4 + s_4 \text{ for Day 4.}$$

$$2x_3 + s_4 = 0.5x_4 + s_5 \text{ for Day 5.}$$

Clearly, the x_j's must be nonnegative. To ensure that we never over spend, the s_j's are required to be nonnegative as well.

5.4 LP Formulation

$$
\begin{array}{ll}
\text{Max} & 2x_4 + s_5 \\
\text{Subject to:} &
\end{array}
$$

$$
\begin{aligned}
s_0 &= x_1 + s_1 \\
s_1 &= 0.5x_1 + x_2 + s_2 \\
2x_1 + s_2 &= 0.5x_2 + x_3 + s_3 \\
2x_2 + s_3 &= 0.5x_3 + x_4 + s_4 \\
2x_3 + s_4 &= 0.5x_4 + s_5
\end{aligned}
$$

$$x_j \geq 0 \text{ for } j = 1, 2, 3, 4 \text{ and } s_j \geq 0 \text{ for } j = 1, 2, 3, 4, 5.$$

This is a linear program with 9 decision variables, 5 functional constraints, and 9 nonnegativity constraints.

5.5 Remarks

1. It may be instructive to attempt to formulate this problem using the x_j's only. Give it a try; it would be quite messy.

2. If we did not realize that x_5 was unnecessary, then the fifth constraint would have come out as $2x_3 + s_4 = 0.5x_4 + x_5 + s_5$. Consider an investment strategy that prescribes, say, $x_5 = 5$ and $s_5 = 10$. We will argue that this strategy cannot be optimal. Observe that the variable x_5 appears only in the fifth constraint. Therefore, if we simply reset x_5 to 0 and s_5 to 15 (maintaining the sum $x_5 + s_5$ at 15), then the resulting new strategy will have a better objective function value (by how much?). This shows that it is never optimal to assign a positive value to x_5.

3. Relax Assumption 2: Suppose instead that there is a reinvestment delay of one day. This implies, for example, that the yield $2x_1$ derived from the new investment on Day 1 won't be available until Day 4. Therefore, we should delete the term $2x_1$ from the left-hand side of the third constraint and transfer this term to that of the fourth constraint. This results in revised constraints $s_2 = 0.5x_2 + x_3 + s_3$ and $2x_1 + s_3 = 0.5x_3 + x_4 + s_4$. After making a similar revision in the fifth constraint, we arrive at the following new formulation:

$$
\begin{array}{ll}
\text{Max} & 2x_4 + s_5 \\
\text{Subject to:} &
\end{array}
$$

$$
\begin{aligned}
s_0 &= x_1 + s_1 \\
s_1 &= 0.5x_1 + x_2 + s_2 \\
s_2 &= 0.5x_2 + x_3 + s_3 \\
2x_1 + s_3 &= 0.5x_3 + x_4 + s_4 \\
2x_2 + s_4 &= 0.5x_4 + s_5
\end{aligned}
$$

$$x_j \geq 0 \text{ for } j = 1, 2, 3, 4 \text{ and } s_j \geq 0 \text{ for } j = 1, 2, 3, 4, 5.$$

   Note that it is not necessary to revise the objective function.

4. Relax Assumption 3: Suppose the daily interest rate is 1%. Then, our (original) formulation revises to:

$$
\begin{array}{ll}
\text{Max} & 2x_4 + 1.01s_5 \\
\text{Subject to:} &
\end{array}
$$

$$
\begin{aligned}
s_0 &= x_1 + s_1 \\
1.01s_1 &= 0.5x_1 + x_2 + s_2 \\
2x_1 + 1.01s_2 &= 0.5x_2 + x_3 + s_3 \\
2x_2 + 1.01s_3 &= 0.5x_3 + x_4 + s_4 \\
2x_3 + 1.01s_4 &= 0.5x_4 + s_5
\end{aligned}
$$

$$x_j \geq 0 \text{ for } j = 1, 2, 3, 4 \text{ and } s_j \geq 0 \text{ for } j = 1, 2, 3, 4, 5,$$

where every daily saving grows by a factor of 1.01 overnight.

5. Relax Assumption 4: If borrowing is allowed, we can simply remove the nonnegativity requirements for the s_j's. For realism, we should, however, introduce some limits on loans, and incorporate that into the formulation in the form of additional constraints. Otherwise, one would attempt to borrow an infinite amount of money.

6 A Project Scheduling Formulation

Nathan and his roommates wake up late on Sundays and clean up their apartment as fast as possible to catch up with their friends for a ball game in the park. There are 8 cleaning steps. Some steps can be started only after some others are finished, the order among these steps is governed by **precedence relations**. For example, sink can only be cleaned after the dishes are washed. Predecessors and duration of each step is listed below:

Predecessor	Step	Duration (mins)
N/A	A = Clean the Fridge	10
N/A	B = Wash the Dishes	25
N/A	C = Make up the Beds	15
A , B	D = Clean the Sink	7
C , D	E = Take the Dust	18
E	F = Vacuum the Carpet	12
D	G = Take the Garbage out	3
F , G	H = Tidy up the Apartment	14

The question is how fast Nathan and his roommates can finish the cleaning while respecting the precedence relations. We make the following assumptions :

1. Nathan has plenty roommates (WLOG, say 8 people) to be assigned to the cleaning steps.

2. The number of people assigned to a step does not affect the duration of that step.

Thus, the work assignment is not the issue. We can focus on the timing of each step.

6.1 Decision Variables

We want to minimize the finishing time of the last step. Since H does not appear as a predecessor to any step, it is the last step. H is completed 14 minutes after its starting time (Assumptions 1 and 2). Thus, the starting times of the cleaning steps are sufficient to characterize the finishing times of all (including the last) step. Let

- t_j = Start time of step j, $j \in \{A, B, C, D, E, F, G, H\}$.

We assume that the first activity starts at time 0.

6.2 Objective Function

We want to minimize the finishing time of the last step:

$$\text{Min } t_H + 14.$$

6.3 Constraints

Because of the precedence relations, a step can only start after its predecessor is finished. We will use a single inequality constraint to represent each precedence relation. For example, "sink can only be cleaned after the dishes are washed" or B precedes D is represented as:

$$\text{Constraint B} \rightarrow \text{D} : t_D \geq t_B + 25$$

Similarly "sink can only be cleaned after the fridge is cleaned" or A precedes D is written as:

$$\text{Constraint A} \rightarrow \text{D} : t_D \geq t_A + 10$$

For each remaining precedence relation, we write a constraint:

$$\text{Constraint C} \rightarrow \text{E} : t_E \geq t_C + 15$$

$$\text{Constraint D} \rightarrow \text{E} : t_E \geq t_D + 7$$

$$\text{Constraint E} \rightarrow \text{F} : t_F \geq t_E + 18$$

$$\text{Constraint D} \rightarrow \text{G} : t_G \geq t_D + 7$$

$$\text{Constraint F} \rightarrow \text{H} : t_H \geq t_F + 12$$

$$\text{Constraint G} \rightarrow \text{H} : t_H \geq t_G + 3$$

On top of these 8 functional constraints, we add nonnegativity constraints: $t_A, t_B, t_C \geq 0$.

6.4 Remarks

1. Convince yourself that the nonnegativity constraints on D, E, F, G, H are not needed. These constraints are *implied* by functional constraints and the nonnegativity constraints on A, B and C.

2. Does the optimal solution change if we drop 14 from the objective function? How about the objective of Min $2t_H$, does the optimal solution change this time? Any generalizations?

3. Problems of this type are known as "activity scheduling" problems. Besides LP, a method called CPM (critical path method) can be used to obtain a solution.

7 Exercises

1. A furniture manufacturer produces and sells TV stands at a price of \$100. If 10 or fewer stands are manufactured per week, each stand costs \$60. Manufacturer's regular capacity is 10 stands per week. Manufacturer can hire additional workforce to bring its capacity up to 20 stands per week. However, additional capacity is costly so any stand produced after the 10th costs \$75. For example 12 stands cost $10 \cdot 60 + 2 \cdot 75 = 750$ dollars. Manufacturer has a weekly operating capital of \$1200 so its weekly costs can not exceed this amount. It also has committed to deliver to a customer 2 stands every week. Provide an LP formulation to maximize the manufacturer's profit.

2. A young college professor Dr.Martin decides to supplement his income by raising chicken and rabbits in his balcony. Each rabbit sells at \$25 at the farmer's market and the price for a chicken is \$15. Rabbits (chickens) are sold 20 (18) weeks after their birth. Dr.Martin has a 12 m^2 balcony and each rabbit (chicken) needs 2 (0.5) m^2 living area. Raising a rabbit (chicken) costs \$1 (\$0.6) (these are mainly food costs) and Dr.Martin has only \$5 per week to spend on rabbit/chicken food.
a) Provide an LP formulation to maximize Dr.Martin's revenue from rabbit and chicken sales.
b) Dr.Martin realizes that his chickens are too aggressive and are attacking young rabbits. He decides to raise at least four times as many rabbits as chickens so that rabbits can defend themselves. Modify the formulation to reflect this restriction.

3. Because of the poor air quality in Dallas (partly due to too many private cars in the traffic), DART wants to start up 4 new routes: between Plano and Dallas, between Richardson and Addison, between Plano and Richardson, and between Richardson and Dallas. Plano and Dallas municipalities put down \$800,000 and \$200,000, respectively, to finance these routes. DART is facing the question of investing a total of \$1,000,000 to four routes such that at least 80% of the investment is made for the routes involving Plano and at least 30% of the investment is made for the routes involving Dallas. Richardson - Addison and Plano - Richardson routes are almost of equal length and DART can make a profit at the rate of 20% per year for each dollar invested. These rates are 0% for the Plano - Dallas route and 10% for the Richardson - Dallas route.
   a) Give an LP formulation that maximizes DART's return on investment in a year.
   b) Argue that there can be an optimal solution without investing into Richardson - Addison route.

4. Refer to Production Planning Example. Suppose we are modelling cheese production and cheese can be kept at most a month in storage before consumption. If it is produced in Jan, it can be kept in the inventory during Feb and can be used to meet Feb demand. But that cheese cannot be used to meet Mar demand. We also suppose that 3 Assumptions made for production planning example are still valid. Modify the production planning LP to model cheese production.
   Some of you may think that in this case the inventory at the end of month j must always be $x_j - d_j$. We illustrate that this is not necessarily so with the following example. Say $d_1 = 30$,$d_2 = 50$, $x_1 = 50$ and $x_2 = 40$. Starting with $y_0 = 0$, $y_1 = 20$. In Feb, we use y_1 to meet 20 of 50 units of demand and use x_2 to meet remaining 30 of 50 units of demand. Eventually, $y_2 = 10$ units of x_2 is passed to Mar as inventory. But

   $$10 = y_2 \neq x_2 - d_2 = -10 \qquad \text{while} \qquad 10 = y_2 = x_1 - d_1 + x_2 - d_2 = 20 + (-10).$$

   Thus, saying $y_j = x_j - d_j$ is an incorrect answer.

5. Textbook H-L: p. 95 3.4.9)a., pp. 96-97 3.4.13)a. and b.

6. Suppose that x_i denotes decision variables, which of the following mathematical relationships can be found in an LP:

   (a) $x_1 - 2x_2 + 3x_4 \geq -8$
   (b) $3x_1 - x_2x_3 = 3$
   (c) $(x_2 + x_3)/x_1 \leq -7$
   (d) $x_1^2 + x_2^2 - x_3^2 = 0$
   (e) $x_1^{x_2} = 2$
   (f) $sin(\pi/2)x_1 + 10^8x_2 = 1$
   (g) $x_2^3 = 8$

   Which of these relationships can be reduced to an equivalent form so that the new form can be put into an LP?
   When you are attempting to convert a nonlinear function into a linear function, you should not change variables. Let us illustrate with an example. Clearly $f(x_1, x_2) = x_1^2 - x_2$ is a nonlinear function of x_1. You can set $y := x_1^2$ to obtain $f(\sqrt{y}, x_2) = y - x_2$ and you may define $g(y, x_2) := f(\sqrt{y}, x_2)$. Then g is a linear function of y and x_2. But the actual question is whether f is linear in x_1!

7. TexOil Company can buy two types of crude oil: light oil at \$20/barrel and heavy oil \$17/barrel. When a barrel of oil is refined it yields gasoline, jet fuel and kerosene in the following quantities (in terms of barrel of output per barrel of input):

	Gasoline	Jet Fuel	Kerosene
Light oil	0.43	0.20	0.28
Heavy oil	0.32	0.38	0.21

TexOil has promised to deliver 800,000 barrels of gasoline, 1,000,000 barrels of Jet Fuel and 300,000 barrels of Kerosene. Provide an LP formulation that keeps promises and minimizes total oil purchase cost.

8. PlaCar company manufactures light truck and SUV bodies. The production requires certain amount of steel and labor as shown below with the availability and cost information:

	Steel (kgs)	Labor (hrs)
Truck body	1480	18
SUV body	1820	15
Unit cost ($)	3	12
Total available	1,000,000	15000

According to forecasts at most 800 Truck bodies at $6000 each and 650 SUV bodies at $7200 each can be sold.
a) Provide an LP to maximize PlaCar's profit.
b) Big vehicles consume too much gas, suppose that EPA (Environmental Protection Agency) charges a regulatory fine to body manufacturers of $500 for each body that weighs more than 1500 kgs. How would you modify your formulation in a).
c) Light truck bodies are narrower than SUV bodies. To bring down the required amount of steel for SUVs down to 1480kgs and to avoid EPA's fine, PlaCar decides to use a slight modification of light truck body for its SUVs. How would you modify your formulation in a).
d) Given the formulation in c), can we deduce that PlaCar will produce 650 SUVs and several trucks, explain.

9. Textbook H-L: p.98-99 3.4-19)a. (Do not do part b).

10. Solve H-L p.98, 3.4.17 a)

11. Loan Portfolio Model: The Bank of America is formulating a loan policy to utilize a maximum of $100 millionofits assets. The following table provides data for different types of loans: Unrecoverable

Loan	Interest	Chance of no recovery
Farm	0.05	0.01
Home	0.08	0.03
Car	0.12	0.05
Grad Education	0.14	0.08
Venture capital	0.15	0.20

loans do not generate interest income. Bank of America wants to dedicate at least 30% of its loans for education but the chance of no recovery for (weighted average by loan sizes of) the loan portfolio should not exceed 0.06. Formulate an LP to maximize interest income.

12. A Marketing Model: A phone survey company needs to survey 100 wives, 90 husbands, 80 single adult males and 70 single adult females. Making a daytime phone call costs 1.5 dollars per call, the same number is 3 for nightime calls. The likelihood of the type of the person who would pick up the phone during the day and the night are different but given by the following table:

Provide an LP formulation to complete this survey at minimum cost.

	Wife	Husband	Single adult male	Single adult female	Total
Day time	0.25	0.15	0.3	0.3	1.0
Night time	0.25	0.35	0.15	0.25	1.0

13. Linear Regression with Linear Programs: Given data points $\{(x_i, y_i) : i = 1..N\}$, Linear Regression fits the line $y = ax+b$ to data points by minimizing the square of the distance between the data points and the line:

$$\text{Min} \quad \sum_{j=1}^{N}(y_j - ax_j - b)^2.$$

We define the rectilinear distance along the y axis and between two points $e = (e_x, e_y)$ and $f = (f_x, f_y)$ as $|e_y - f_y|$ where e_x, f_x and e_y, f_y are the x, y coordinates of the point e and f respectively. Note that this distance computes distances between two points only along the y axis (i.e. $|e_y - f_y|$) and disregards the distance along the x axis (i.e. $|e_x - f_x|$). For example, if $e = (3, 7)$ and $f = (4, 5)$, the rectilinear distance along y is $|e_y - f_y| = 2$.
a) Formulate an LP to find a line $y = ax + b$ closest (in rectilinear distance along y) to a given set of data points $\{(x_i, y_i) : i = 1..N\}$. You may assume that $N = 3$ if you are not comfortable with summations and indexing.
b) Find the optimal solution (the objective value and the equation of the line) by inspection when $N = 2$.

14. Manpower Planning: At the post office on the Coit street, each employee works exactly for 5 **consecutive** days per week. To provide a satisfactory customer service, the post office needs the following number of employees each day:

Days	Mon	Tue	Wed	Thu	Fri	Sat	Sun
# of employees required	9	6	5	8	11	13	4

a) Provide an LP formulation to minimize the number of employees.
b) Suppose some employees are willing to do overtime and work for one more day right after their 5 day regular schedule. Suppose that overtime labor rate is 50% more than regular labor rate. Provide an LP formulation to minimize the labor costs.

15. Market clearing model: Consider a market with I suppliers, J buyers and K products being bought and sold. A generic supplier is named as supplier i, a generic buyer is named buyer j and a generic product is named product k. Supplier i supplies at most S_{ik} units of product k and asks $\$A_{ik}$ price per unit of product k. Buyer j demands at most D_{jk} units of product k and bids $\$B_{jk}$ per unit of product k. Suppose that you work as an auction: You collect all the pricing and bidding information, then allocate supplies to buyers. This mechanism is called market clearing. You would be making profit from the spread between the bidding price B_{jk} and the asking price A_{ik} for every unit of product k sold from supplier i to buyer j.
a) Provide an LP formulation to maximize market clearing profit.
b) Suppose supplier i and buyer j are not at the same location and the available fleet capacity allows for transportation of at most U_{ij} products from supplier i to buyer j. In addition you incur a cost of $\$C_{ij}$ per unit of any product transported from supplier i to buyer j. Provide an LP formulation to maximize market clearing profit under these additional conditions.
c) Suppose that the fleet capacities U_{ij} of b) are given in terms of kilos and each product k weighs W_k kilos. Modify your formulation to b).

16. Road construction model: Suppose that we draw a line from the intersection of Coit & G. Bush to the intersection of Route 75 & G. Bush. Let h_i be the constant and known elevation (from the sea level)

of the terrain on the line from the $i-1$st kilometer to the ith kilometer measured from the intersection of Coit & G. Bush, for $i = 1..N$. As a test of understanding the notation, note that N is the distance in kilometers between intersections and h_1 is the constant elevation of the line while traversing it from the intersection of Coit & G. Bush towards Route 75 for 1 kilometer. Further suppose that you want to construct a road between these two intersections by adding or removing terrain so as to make your road smooth. Actually, it is required that the slope of your road (in either direction) can not be larger than s. If you are willing to add and remove a lot of terrain, you can actually construct a level road all the way, needless to say this would require a huge budget. Provide an LP that will minimize the total terrain added or removed for the road construction and respect the slope constraint.

8 Solutions of Selected Problems

1. Let x = number of TV stands produced in a week. Let c = cost of producing x TV stands per week. I suggest that you draw the cost function c as x varies. You will see that c is composed of two lines intersecting at $x = 10$. The equation of the lines are $c = 60x + 0$ and $c = 75x - 150$. The first line gives the cost for $x \leq 10$ and the second gives the cost for $x \geq 10$. Convince yourself that the cost can be written as $c = \max\{60x,\ 75x - 150\}$. In the formulation below we represent the maximum with inequalities, note that those inequalities are similar to the ones in the *Nathan's apartment cleaning example.*

$$
\begin{array}{llrrcrl}
\text{Max} & & 100x & -c & & & \\
\text{Subject to:} & & & & & & \\
 & & 60x & -c & \leq & 0 & (1) \\
 & & 75x & -c & \leq & 150 & (2) \\
 & & & c & \leq & 1200 & (3) \\
 & & x & & \geq & 2 & (4) \\
 & & x, & c & \geq & 0 & .
\end{array}
$$

2. a) Suppose that Dr. Martin always keeps the same number of rabbits and chickens. Let r (c) = number of rabbits (chicken) raised at any time. The revenue obtained by raising a rabbit for a week is \$25/20 and the same number is \$15/18 for a chicken.

$$
\begin{array}{llrrcrl}
\text{Max} & & (25/20)r & +(15/18)c & & & \\
\text{Subject to:} & & & & & & \\
 & & 2r & +0.5c & \leq & 12 & (1) \\
 & & r & +0.6c & \leq & 5 & (2) \\
 & & r, & c & \geq & 0 & .
\end{array}
$$

b) Add $r - 4c \geq 0$.

3. a) Let x_{ij} = dollar investment for route ij where i and j are the initials of Dallas, Plano, Addison or Richardson.

$$
\begin{array}{llrrrrcrl}
\text{Max} & & x_{PD} & +1.2x_{RA} & +1.2x_{PR} & +1.1x_{RD} & & & \\
\text{Subject to:} & & & & & & & & \\
 & & x_{PD} & +x_{RA} & +x_{PR} & +x_{RD} & = & 1,000,000 & (1) \\
 & & x_{PD} & & +x_{PR} & & \geq & 800,000 & (2) \\
 & & x_{PD} & & & +x_{RD} & \geq & 300,000 & (3) \\
 & & x_{PD}, & x_{RA}, & x_{PR}, & x_{RD}, & \geq & 0 & .
\end{array}
$$

b) Let $\bar{x}_{RA} > 0$ and $\bar{x}_{PR}$ be the value of optimal investment for Richardson - Addison route and Plano - Richardson route. Consider another solution where we modify only x_{RA} and x_{PR} as $x_{RA} = 0$ and

$x_{PR} = \bar{x}_{PR} + \bar{x}_{RA}$. The new solution is feasible (why?) and has the same objective value. Thus, the new solution is optimal and it does not require investing for Richardson - Addison route.

9 LP Problems and Solutions from Plano High School

1. During slack time, the sawing and fabricating benches in Milam Cabinet Shop are used to make wooden hanging baskets and plant stands. The sawing bench has at most ten hours, and the fabricating bench has at most eight hours of slack time each week. Hanging baskets take one-fourth hour of sawing, while plant stands take one-third hour of sawing. Hanging baskets take one-half hour of fabricating, but plant stands take one-sixth hour of fabricating. Hanging baskets sell for a profit of $6, and plant stands sell for a profit of $8. Write a mathematical model to maximize profits.

   H = # of Hanging baskets
   P = # of Plant stands
   Max 6H + 8P
   s.t. $\frac{1}{4}H + \frac{1}{3}P \leq 10$
   $\frac{1}{2}H + \frac{1}{6}P \leq 8$
   $H \geq 0$
   $P \geq 0$

2. The Holland family decides to raise and sell peppers and tomatoes to supplement their income. They have six acres of land. They believe they can make $2,000 an acre on peppers and $3,000 an acre on tomatoes. From past experience, they feel that they cannot take care of more than five acres of peppers or four acres of tomatoes. Write a mathematical model to show how many acres of each they should grow to maximize profit.

   P = # of acres of peppers
   T = # of acres of tomatoes
   Max 2000P + 3000T
   s.t. $P \leq 5$
   $T \leq 4$
   $P + T \leq 6$
   $P \geq 0$
   $T \geq 0$

3. Bonham Garden Fertilizers produce Regular and Super-Gro formulations. There are ten employees or 400 hours of production time each week. It takes one-fourth hour to produce and package either Regular or Super-Gro. Bonham has $7,000 to spend on raw materials. Raw materials cost $2 package for Regular, and $5 per package for Super-Gro. Bonham makes $1 profit on Regular and $2 profit on Super-Gro per package. Write a mathematical model to show how many packages of each Bonham should formulate to maximize profit.

   R = # of packages of Regular
   S = # of packages of Super-Gro
   Max R + 2S
   s.t. $2R + 5S \leq 7000$
   $\frac{1}{4}(R + S) \leq 400$
   $R \geq 0$

$S \geq 0$

4. Thomas & Brown Accounting audits books and prepares tax return. It employs three CPAs, each working 40 hours per week. The owners, after looking for new clients, have a total of 20 hours per week to review the work. The profit on an audit is $400 and on a tax return $125. An audit requires 6 hours of CPA time and 2 hours of review time. A tax return requires 3 hours of CPA time and one-fourth hour of review time. Write a mathematical model to show what mix of tasks they should do to maximize profits.

   A = # of audits
   T = # of tax returns
   Max 400A + 125T
   s.t. $6A + 3T \leq 120$
   $2A + \frac{1}{4}T \leq 20$
   $A \geq 0$
   $T \geq 0$

5. Jim Fannin Pharmaceuticals sells to drug stores. As an independent jobber, he can choose his own territory. A call on a small-town drug store usually results in a $1,200 sale, but takes an average of six hours. A call on a big-city drug store usually results in a $750 sale but only takes an average of two hours. Jim doesnt work more than 42 hours a week. Write a mathematical model to show how he should spend his time calling on customers to maximize sales.

   S = # of calls in small town
   B = # of calls in big city
   Max 1200S + 750B
   s.t. $6S + 2B \leq 42$
   $S \geq 0$
   $B \geq 0$

6. The Biltrite Furniture Company makes wooden desks and chairs. Carpenters and finishers work on each item. On the average the carpenter spends 4 hours working on each chair and 8 hours working on each desk. There are enough carpenters for at most 8000 worker-hours per week. The finishers spend about 2 hours on each chair and 1 hour on each desk. There are enough finishers for up to 1300 worker-hours per week. If there is a profit of $80 per chair and $135 per desk what production level will maximize the profit?

   C = # of chairs produced
   D = # of desks produced
   Max 80C + 135D
   s.t. $4C + 8D \leq 8000$
   $2C + D \leq 1300$
   $C \geq 0$
   $D \geq 0$

7. Jocelyns Jewelry Store makes rings and pendants. Every week the staff uses at most 500 gm of metal and spends at most 150 hours making jewelry. It takes 5 gm of metal to make a ring and 20 gm to

make a pendant. Each ring takes 2 hours and each pendant requires 3 hours to make. The profit on each ring is $70 and the profit on each pendant is $90. How many of each should the store make to maximize its profit?

R = # of rings made
P = # of pendants made
Max 70R + 90P
s.t. $5R + 20P \leq 500$
$2R + 3P \leq 150$
$R \geq 0$
$P \geq 0$

8. Major Motors must produce at least 5,000 luxury cars and 12,000 medium-prices cars. They must also produce at most 30,000 compact cars. The company owns two factories A and B at different locations. Factory A produces 20, 40 and 60 units of luxury, medium and compact cars, respectively each day. Factory B produces 10, 30 and 50, respectively, each day. If factory A costs $960,000 per day to operate and B costs $750,000 per day, find the number of days each should operate to minimize the costs yet meet the requirements for car production. What is the minimum cost?

A = # of days factory A operates
B = # of days factory B operates
Min 960000A + 750000B
s.t. $20A + 10B \geq 5000$
$40A + 30B \geq 12000$
$60A + 50B \leq 30000$
$A \geq 0$
$B \geq 0$

9. Dr. Delphinium Gardening Supplies contracts, on a weekly basis, for suppliers for its stores. Clay Corner can provide 150 glazed and 400 unglazed flowerpots per week. It can commit to at most 15 weeks of production. The contract is for $500 per week. Wheel Works can provide 50 glazed and 100 unglazed flowerpots per week. It can commit to at most 35 weeks of production. This contract is for $250 per week. To satisfy existing orders for spring shipment of plants, Dr. Delphinium needs at least 2250 glazed and 5000 unglazed flowerpots. How many weeks of production from each supplier should be contracted to minimize costs? What is the minimum cost?

C = # of weeks of production from Clay Corner
W = # of weeks of production from Wheel Works
Min 500C + 250W
s.t. $150C + 50W \geq 2250$
$400C + 100W \geq 5000$
$C \leq 15$
$W \leq 35$
$C \geq 0$
$W \geq 0$

10. Ruby Sappire Culpepper, who is into being fit, takes vitamin pills. Each day she must have at least 16 units of vitamin A, and at least 5 units of vitamin B_1, and at most 20 units of vitamin C. She can

choose between pill 1 which contains 8 units of A, 1 unit of B_1, and 2 units of C. Pill 2 contains 2 units of A, 1 unit of B_1, and 2 units of C. Pill 1 costs 15 cents and pill 2 costs 30 cents. How many of each pill should she buy in order to minimize her cost? What is that cost?

X = # of pill 1 she buys
Y = # of pill 2 she buys
Min 15X + 30Y
s.t. $8X + 2Y \geq 16$
$X + Y \geq 5$
$2X + 2Y \leq 20$
$X \geq 0$
$Y \geq 0$