The Graphical Simplex Method: An Example

Consider the following linear program:

$$
\begin{array}{llrrcll}
\text{Max} & & 4x_1 & +3x_2 & & & \\
\text{Subject to:} & & & & & & \\
& & 2x_1 & +3x_2 & \leq & 6 & (1) \\
& & -3x_1 & +2x_2 & \leq & 3 & (2) \\
& & & 2x_2 & \leq & 5 & (3) \\
& & 2x_1 & +x_2 & \leq & 4 & (4) \\
& & & x_1,\ x_2 & \geq & 0\ . &
\end{array}
$$

Goal: produce a pair of x_1 and x_2 that (i) satisfies all constraints and (ii) has the greatest objective-function value.

A pair of specific values for (x_1, x_2) is said to be a **feasible solution** if it satisfies all the constraints.

$(x_1, x_2) = (0, 0)$ and $(x_1, x_2) = (1, 1)$ are feasible. $(x_1, x_2) = (1, -1)$ and $(x_1, x_2) = (1, 2)$ are not feasible. The objective-function value at $(0, 0)$ is 0 and at $(1, 1)$ is 7.

The Graphical Simplex Method: An Example

(x_1, x_2) is a point in the coordinate system.

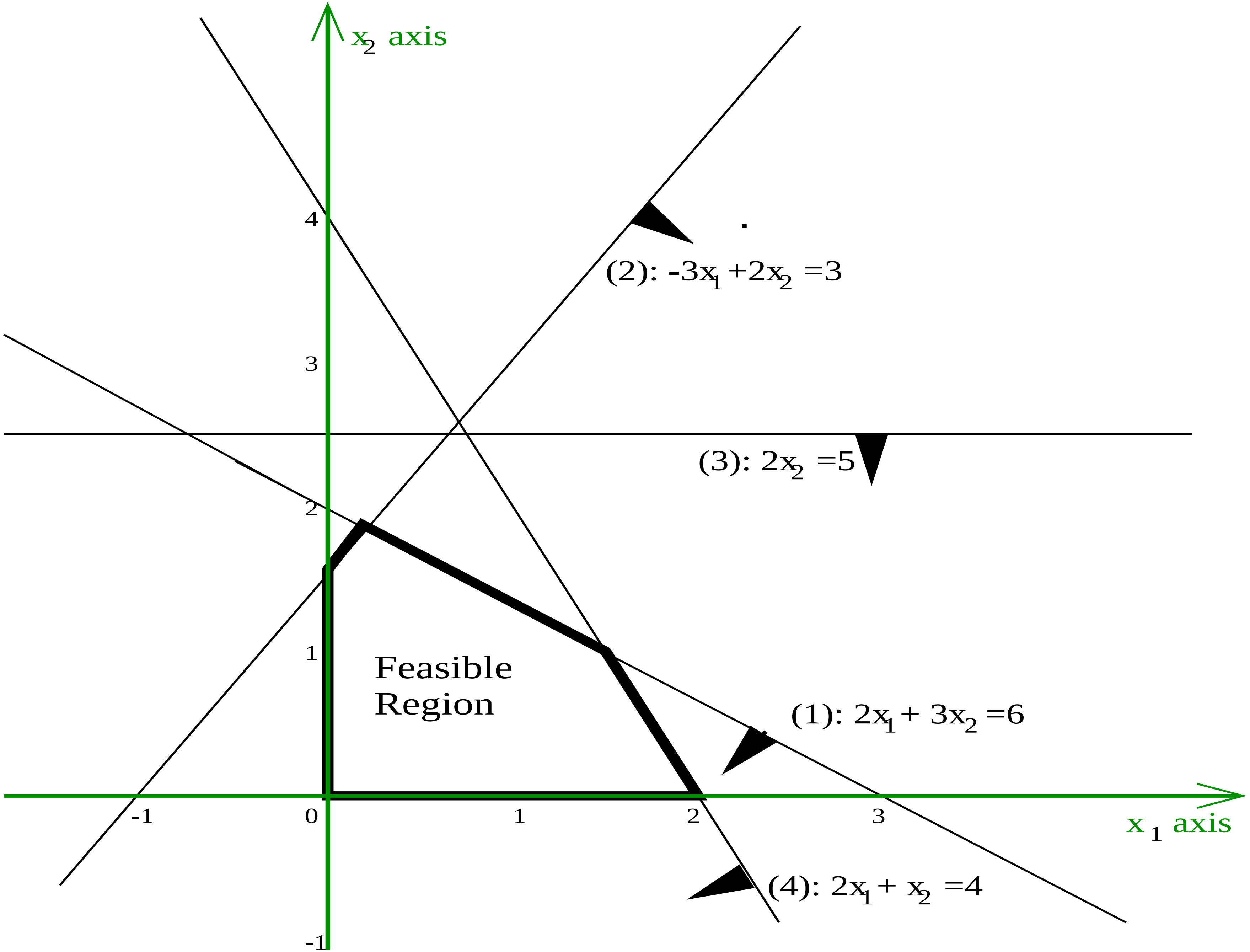
Let us turn inequalities into equalities and draw lines on the coordinate system.

Observe that each line (1) the plane into two half-planes: Feasible half and infeasible half. We indicate the feasible half with arrows.

Draw other lines (2), (3) and (4) and indicate the feasible half for all the lines.

The region that is on the correct side of all lines: **Feasible region** or **feasible set**?

Note that constraint (3) is *redundant*.


x2 axis
4
3
2
1
-1
-1
0
1
2
3
x 1 axis
(2): -3x1 +2x2 =3
(3): 2x2 =5
(1): 2x1 + 3x2 =6
(4): 2x1 + x2 =4
Feasible
Region

The Graphical Simplex Method: An Example

Optimality?

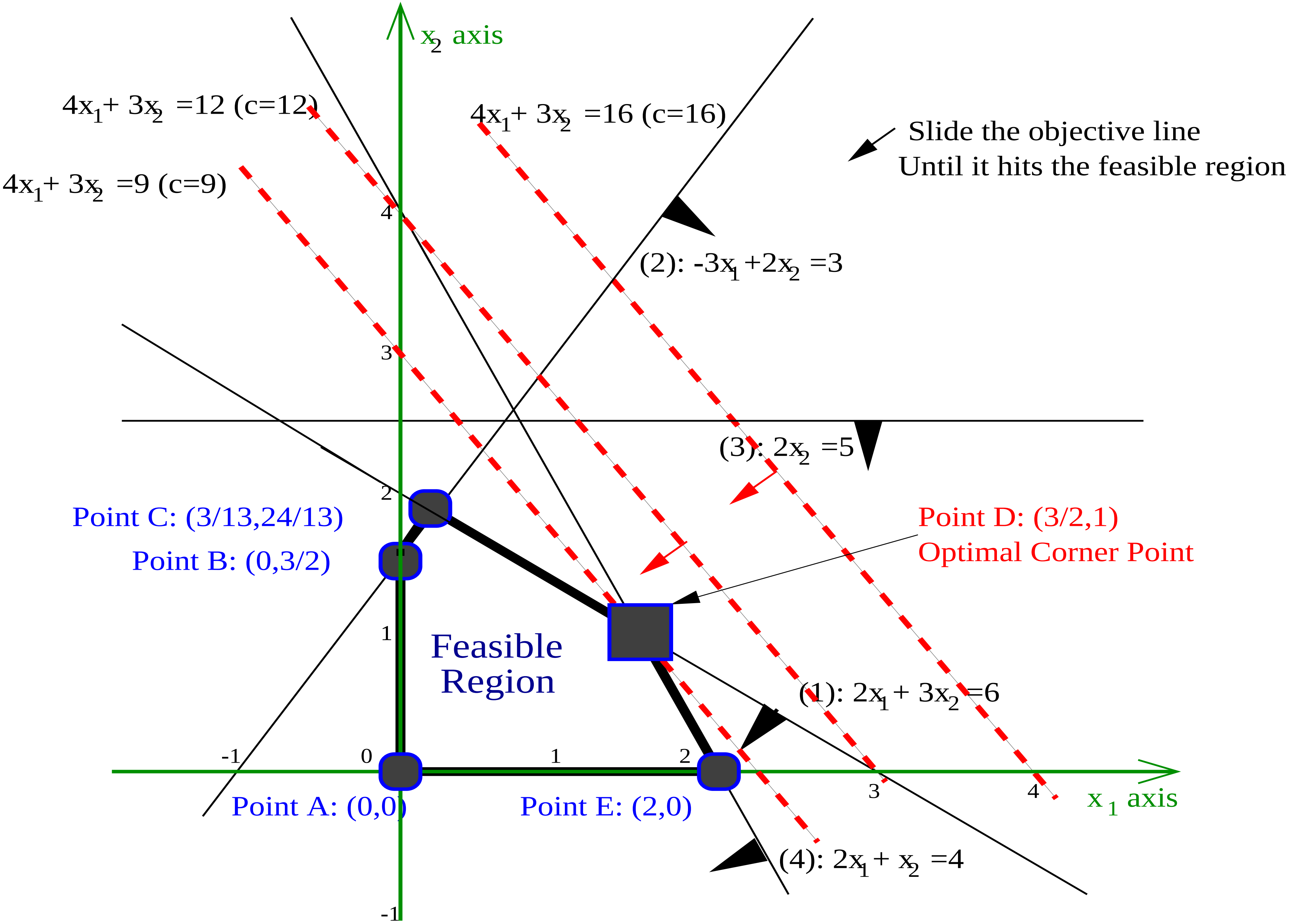
For any given constant c, the set of points satisfying $4x_1 + 3x_2 = c$ is a straight line.

By varying c, we can generate a family of lines with the same slope.

The line with the smaller c is closer to the feasible region $\Longrightarrow$ Decrease c further to reach the feasible region.

Corner point (or an **extreme point**) of the feasible region ?

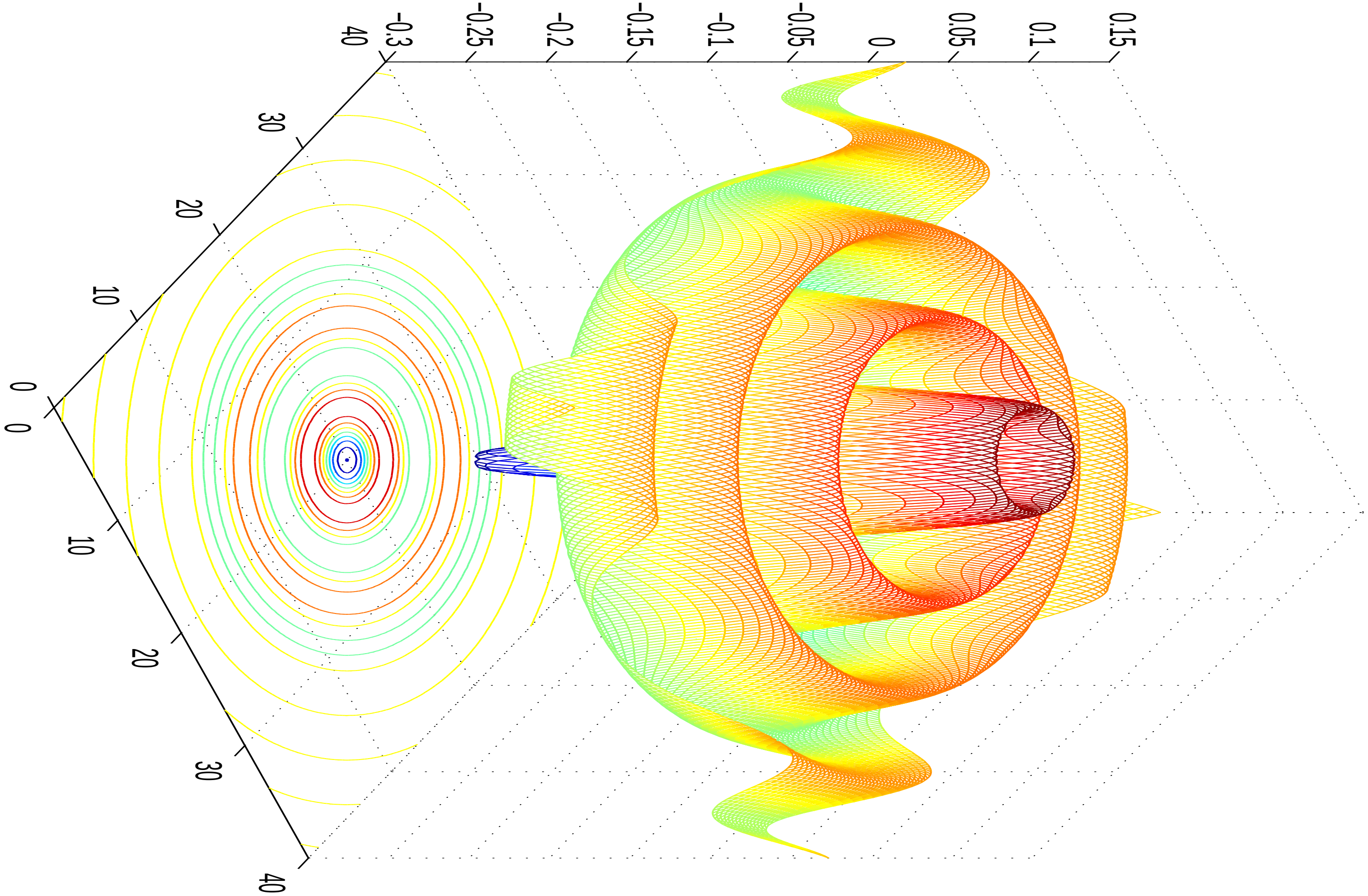
The corner point, marked as D must be optimal. $D = (3/2, 1)$ and has an objective-function value of 9.



x_2 axis
$4x_1 + 3x_2 = 12$ (c=12)
$4x_1 + 3x_2 = 16$ (c=16)
$4x_1 + 3x_2 = 9$ (c=9)
Slide the objective line
Until it hits the feasible region
(2): $-3x_1 + 2x_2 = 3$
(3): $2x_2 = 5$
Point C: (3/13,24/13)
Point B: (0,3/2)
Point D: (3/2,1)
Optimal Corner Point
Feasible Region
(1): $2x_1 + 3x_2 = 6$
Point A: (0,0)
Point E: (2,0)
(4): $2x_1 + x_2 = 4$
x_1 axis

Why do we like Linear Programs but not the Flowers?

$R = \sqrt{(x-20)^2 + (y-20)^2}$ **and** $z = (sin(R+4))/(R+4)$.



Procedure Solve LP

Solving LPs is simple:

1. Identify the coordinates of all corner points of the feasible region.
2. Evaluate the objective function at all of these corner points.
3. Pick the best corner point.

Determining the coordinates of the corner points *without* a graph.

Extreme points are at the intersection of constraints. For example, D is at the intersection of lines (1) and (4). Thus, we solve the system of two **defining equations**:

$$\begin{array}{rrcl} 2x_1 & +3x_2 & = & 6 \qquad (1) \\ 2x_1 & +x_2 & = & 4 \qquad (4) \end{array}$$

$D = (x_1, x_2) = (3/2, 1)$ solves the equations. Need to identify the defining equations for each corner point without the graph.

Procedure Solve LP: Procedure Generate Corner Points

Independence from the graph?

1. From the given set of six equations (including $x_1 = 0$ and $x_2 = 0$), choose an arbitrary combination of two equations. Solve these equations to obtain the coordinates of their intersection.

2. If the solution is feasible, then it is a corner-point solution. Otherwise, discard it.

3. Go to 1 unless all combinations are studied.

This procedure generates the coordinates of all corner-pointsolutions.

Procedure Solve LP: Procedure Generate Corner Points

Does the procedure reject infeasible points? Consider equations (1) and (3):

$$\begin{array}{rrcll} 2x_1 & +3x_2 & = & 6 & \qquad (1) \\ & 2x_2 & = & 5\,. & \qquad (3) \end{array}$$

The solution $(x_1, x_2) = (-3/4, 5/2)$ is not feasible, why? Therefore, equations (1) and (3) do not lead to a corner-point solution.

With 6 equations, it is easily seen that the total number of subsets of 2 equations is

$$\binom{6}{2} = \frac{6!}{2!4!} = 15$$

After cycling through all 15 of these combinations and discarding combinations that do not yield a feasible solution, only five combinations remain.

Procedure Solve LP: Remarks

1. Infeasibility: In general, the feasible region of a linear program may be empty. Procedure Solve LP is meaningful only if the feasible region is not empty. A linear program of this type is said to be **infeasible**.

2. Unboundedness: Consider the linear program: Maximize $x_1 + x_2$, subject to $x_1, x_2 \geq 0$. The feasible region has exactly one corner point, at $(0, 0)$; and that this corner point is not optimal. This clearly is a disturbing factor for Procedure Solve LP. A linear program of this type is said to be **unbounded**.

3. The equation pair
$$\begin{array}{rrcl} x_1 & +2x_2 & = & 3 \\ x_1 & +2x_2 & = & 6 \end{array}$$
has no solution. This implies infeasibility of an LP.

4. The equation pair
$$\begin{array}{rrcl} x_1 & +2x_2 & = & 3 \\ 2x_1 & +4x_2 & = & 6 \end{array}$$
has an infinite number of solutions. This does not imply anything about an LP.

A Characterization of the Corner-Point Solutions

Consider the example of the previous section again. Rewrite that problem as:

$$
\begin{array}{llllllll}
\text{Max} & 4x_1 & +3x_2 & & & & & \\
\text{Subject to:} & & & & & & & \\
& 2x_1 & +3x_2 & +s_1 & & & & = 6 \quad (1) \\
& -3x_1 & +2x_2 & & +s_2 & & & = 3 \quad (2) \\
& & 2x_2 & & & +s_3 & & = 5 \quad (3) \\
& 2x_1 & +x_2 & & & & +s_4 & = 4 \quad (4) \\
& \multicolumn{7}{l}{x_1,\ x_2 \geq 0 \ \text{ and } \ s_1,\ s_2,\ s_3,\ s_4 \geq 0\,,}
\end{array}
$$

Hey, this problem looks more difficult! “Simplifying complication”.

Feasible to original problem is feasible to augmented problem

Equivalence via examples:

Consider the feasible solution $(x_1, x_2) = (1, 1)$ to the original problem

$$\begin{array}{rrrrrr} 2\cdot 1 & +3\cdot 1 & = & 5 & \leq & 6 \\ -3\cdot 1 & +2\cdot 1 & = & -1 & \leq & 3 \\ & 2\cdot 1 & = & 2 & \leq & 5 \\ 2\cdot 1 & +1\cdot 1 & = & 3 & \leq & 4\,, \end{array}$$

Slack: for each constraint the difference between the constant on the rhs and the evaluation on the lhs. Slacks are 1 $(= 6 - 5)$, 4 $(= 3 - (-1))$, 3 $(= 5 - 2)$, and 1 $(= 4 - 3)$, respectively. Let $s_1 = 1$, $s_2 = 4$, $s_3 = 3$, and $s_4 = 1$.

The **augmented solution** $(x_1, x_2, s_1, s_2, s_3, s_4) = (1, 1, 1, 4, 3, 1)$ is a feasible solution to the new problem.

Feasible to augmented problem is feasible to original

Consider the feasible solution $(x_1, x_2, s_1, s_2, s_3, s_4) = (0, 0, 6, 3, 5, 4)$ to the augmented problem.

$$\begin{array}{rrrrrrcl} 2\cdot 0 & +3\cdot 0 & +6 & & & & = & 6 \\ -3\cdot 0 & +2\cdot 0 & & +3 & & & = & 3 \\ & 2\cdot 0 & & & +5 & & = & 5 \\ 2\cdot 0 & +1\cdot 0 & & & & +4 & = & 4\,, \end{array}$$

Drop the last four values in $(0, 0, 6, 3, 5, 4)$ and consider the solution $(x_1, x_2) = (0, 0)$. These four values are nonnegative so

$$\begin{array}{rrcl} 2\cdot 0 & +3\cdot 0 & \leq & 6 \\ -3\cdot 0 & +2\cdot 0 & \leq & 3 \\ & 2\cdot 0 & \leq & 5 \\ 2\cdot 0 & +1\cdot 0 & \leq & 4\,, \end{array}$$

$(0, 0)$ is feasible to the original problem.

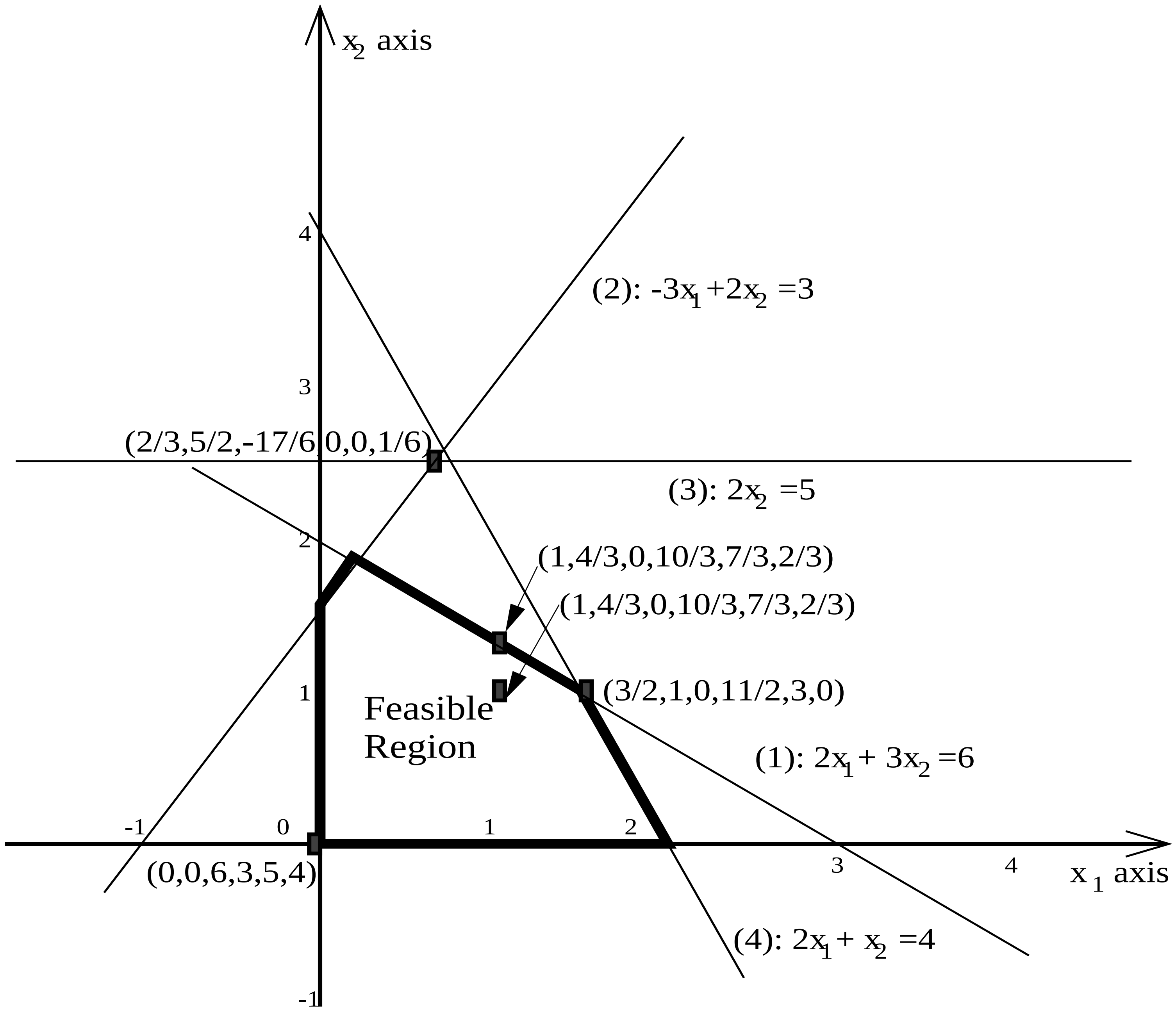
Original problem and the augmented problem are quivalent.

A Characterization of the Corner-Point Solutions

The value of the slack variable provides **explicit** information on the "tightness" of the corresponding original (inequality) constraint.

Consider $(x_1, x_2) = (3/2, 1)$. It has the corresponding augmented solution $(x_1, x_2, s_1, s_2, s_3, s_4)$=$(3/2, 1, 0, 11/2,\ 3, 0)$. It satisfies constraints (1) and (4) as equalities. **Binding constraints**?

Consider $(x_1, x_2) = (1, 4/3)$ and its corresponding augmented solution $(x_1, x_2, s_1, s_2, s_3, s_4) = (1, 4/3, 0, 10/3, 7/3, 2/3)$. $s_1 = 0$ and the other slack variables are all positive; constraint (1) is binding. $(1, 4/3)$ is on an edge of the feasible region.



x2 axis
(2): -3x1 +2x2 =3
4
3
(2/3,5/2,-17/6,0,0,1/6)
(3): 2x2 =5
2
(1,4/3,0,10/3,7/3,2/3)
(1,4/3,0,10/3,7/3,2/3)
1
(3/2,1,0,11/2,3,0)
Feasible
Region
(1): 2x1 + 3x2 =6
-1
0
1
2
(0,0,6,3,5,4)
3
4
x 1 axis
(4): 2x1 + x2 =4
-1

A Characterization of the Corner-Point Solutions

Consider $(x_1, x_2) = (2/3, 5/2)$ and its corresponding augmented solution $(x_1, x_2, s_1, s_2, s_3, s_4) = (2/3, 5/2, -17/6, 0, 0, 1/6)$. Since $s_2 = 0$ and $s_3 = 0$, constraints (2) and (3) are binding. However, $s_1 = -17/6$. It is not feasible.

Consider $(x_1, x_2) = (0, 0)$, with corresponding augmented solution $(x_1, x_2, s_1, s_2, s_3, s_4) = (0, 0, 6, 3, 5, 4)$. Since x_1 and x_2 both equal to 0 and all slack variables are positive, $(0, 0)$ is feasible, and that the nonnegativity constraints $x_1 \geq 0$ and $x_2 \geq 0$ are the only binding constraints. It is a corner point.

Consider $(x_1, x_2) = (1, 1)$ and its corresponding augmented solution $(x_1, x_2, s_1, s_2, s_3, s_4) = (1, 1, 1, 4, 3, 1)$. **Interior point**?

A Characterization of the Corner-Point Solutions

Declaring an equation as a defining equation is the same as assigning a value of zero to its slack variable in the augmented problem.

1. Solutions that make $2x_1 + 3x_2 \leq 6$ binding, let $s_1 = 0$ and consider the augmented solutions of the form $(x_1, x_2, 0, s_2, s_3, s_4)$.

2. Solutions that make the inequality $x_1 \geq 0$ binding. Let $x_1 = 0$ and consider augmented solutions of the form $(0, x_2, s_1, s_2, s_3, s_4)$.

3. Solutions that make both $x_1 \geq 0$ and $2x_1 + 3x_2 \leq 6$ binding, let $x_1 = s_1 = 0$ and consider augmented solutions of the form $(0, x_2, 0, s_2, s_3, s_4)$.

Make the first step INDEPENDENT of the GRAPH:

Revised Step 1: Choose an arbitrary pair of variables in the augmented problem, and assign the value zero to these variables. This reduces the functional constraints in the augmented problem to a set of four equations in four unknowns. Solve this system of equations.

A Characterization of the Corner-Point Solutions: Basic solution

Basic solution?

An augmented solution produced by the Revised Step 1 satisfies all functional constraints in the augmented problem.

How about the nonnegativity constraints?

Revised Step 2: If all of the values in the augmented solution produced by the Revised Step 1 are nonnegative, accept it as a corner-point solution; otherwise, discard the solution.

Basic feasible solution?

Systems of Equations

Consider a system of m linear equations in n unknowns, denoted by x_1, x_2, ..., x_n

$$\begin{array}{ccccccc} a_{11}x_1 & +a_{12}x_2 & + & \cdots & +a_{1n}x_n & = & b_1 \\ a_{21}x_1 & +a_{22}x_2 & + & \cdots & +a_{2n}x_n & = & b_2 \\ & & & \cdots & & & \\ a_{m1}x_1 & +a_{m2}x_2 & + & \cdots & +a_{mn}x_n & = & b_m \end{array}$$

a_{ij}'s and the b_j's are given constants.

Assume that $m \leq n$. Under this assumption, the equation system will typically have an infinite number of solutions.

If we arbitrarily select $n - m$ variables and set their values to zero, then the system will be reduced to a set of m equations in m unknowns. The selected set of $n - m$ variables will be called **nonbasic variables**; and the remaining m variables will be called **basic variables**.

Systems of Equations

In our example, $m = 4$ and $n = 6$. Suppose we declare, for example, s_1 and s_4 as nonbasic variables. Then the reduced equation system is:

$$\begin{array}{rrrrcl} 2x_1 & +3x_2 & & & = & 6 \\ -3x_1 & +2x_2 & +s_2 & & = & 3 \\ & 2x_2 & & +s_3 & = & 5 \\ 2x_1 & +x_2 & & & = & 4\,, \end{array}$$

The (unique) solution is $(x_1, x_2, s_2, s_3) = (3/2, 1, 11/2, 3)$. After adding to this two zeros for the nonbasic variables, we obtain the basic (feasible?) solution $(x_1, x_2, s_1, s_2, s_3, s_4) = (3/2, 1, 0, 11/2, 3, 0)$.

- *Anology between basic (feasible) solutions and sport teams.*
  Team = Basic variables. Bench = Nonbasic variables.

The Simplex Method

Define the objective value $z{=}4x_1 + 3x_2$ and Max z. Put $z = 4x_1 + 3x_2$ into constraints.

Maximize z

Subject to:

$$\begin{array}{cccccccccl} z & -4x_1 & -3x_2 & & & & & = & 0 & (0) \\ & 2x_1 & +3x_2 & +s_1 & & & & = & 6 & (1) \\ & -3x_1 & +2x_2 & & +s_2 & & & = & 3 & (2) \\ & & 2x_2 & & & +s_3 & & = & 5 & (3) \\ & 2x_1 & +x_2 & & & & +s_4 & = & 4 & (4) \end{array}$$

$$x_1,\, x_2,\, s_1,\, s_2,\, s_3,\, s_4 \geq 0\,,$$

The new variable z is unrestricted in sign. **standard form?**

Construct an initial basic feasible solution. Each basic feasible solution has 2 nonbasic variables and 4 basic variables. Which 2 are nonbasic variables?

The Simplex Method

Set x_1 and x_2 as nonbasic. Then at least the solution of equations is simple.

$$\begin{array}{cccccc} s_1 & & & & = & 6 \\ & s_2 & & & = & 3 \\ & & s_3 & & = & 5 \\ & & & s_4 & = & 4 \end{array}$$

Solving these equations is trivial. The resulting augmented solution, $(x_1, x_2, s_1, s_2, s_3, s_4) = (0, 0, 6, 3, 5, 4)$ is a basic feasible solution. Let it be our **starting basic feasible** solution.

The value of z associated with this starting basic feasible solution? None of the current basic variables, s_1, s_2, s_3, s_4, appears in equation (0), $z = 0$.

The set of basic variables in a solution is **basis**. What is the basis for $(x_1, x_2, s_1, s_2, s_3, s_4) = (0, 0, 6, 3, 5, 4)$?

Let us go to B

The current basic feasible solution optimal? Imagine yourself standing at point A and attempt to travel toward the direction of either point B or point E.

Consider point B first. To "Travel along the x_2 axis, increase x_2 from its current value 0 to δ.

To maintain feasibility we must also readjust the values of the current basic variables, s_1, s_2, s_3, and s_4.

$$\begin{array}{rrrrrrcl} 2\cdot 0 & +3\cdot(0+\delta) & +s_1 & & & & = & 6 \\ -3\cdot 0 & +2\cdot(0+\delta) & & +s_2 & & & = & 3 \\ & 2\cdot(0+\delta) & & & +s_3 & & = & 5 \\ 2\cdot 0 & +(0+\delta) & & & & +s_4 & = & 4 \end{array}$$

s_1 must assume the new value $6 - 3\delta$.

$$s_2 = 3 - 2\delta \text{ , } s_3 = 5 - 2\delta \text{ and } s_4 = 4 - \delta$$

The new augmented solution $(x_1, x_2, s_1, s_2, s_3, s_4) = (0, \delta, 6-3\delta, 3-2\delta, 5-2\delta, 4-\delta)$.

Let us go to B

Because of nonnegativity, we require that $6-3\delta \geq 0$, $3-2\delta \geq 0$, $5-2\delta \geq 0$, and $4-\delta \geq 0$. The value of δ should not exceed

$$\min\left[\frac{6}{3}, \frac{3}{2}, \frac{5}{2}, \frac{4}{1}\right] = \frac{3}{2}.$$

With $\delta = 3/2$, the new augmented solution is $(x_1, x_2, s_1, s_2, s_3, s_4) = (0, 3/2, 3/2, 0, 2, 5/2)$. Since $s_2 = 0$, the second inequality constraint becomes binding. What if $\delta > 3/2$?

Objective value, from equation (0)

$$z \quad -4 \cdot 0 \quad -3 \cdot (0+\delta) \quad = \quad 0\,.$$

z value will increase from 0 to $3\delta = 9/2$. The value of z goes up at a rate of 3 per x_2.

Let us go to E

Consider point E now. Consider a small increase of size δ for x_1. Revise basic variables:

$$s_1 = 6 - 2\delta \text{ , } s_2 = 3 + 3\delta \text{ , } s_3 = 5 - 0 \cdot \delta \text{ and } s_4 = 4 - 2\delta.$$

The new augmented solution is $(x_1, x_2, s_1, s_2, s_3, s_4) = (\delta, 0, 6-2\delta, 3+3\delta, 5, 4-2\delta)$.

Because of nonnegativity requirements imply that δ should not exceed

$$\min\left[\frac{6}{2}, \frac{4}{2}\right] = 2$$

With $\delta = 2$, the new augmented solution is $(x_1, x_2, s_1, s_2, s_3, s_4) = (2, 0, 2, 9, 5, 0)$. Since $s_4 = 0$, the fourth inequality constraint becomes binding. What if $\delta > 2$?

Objective value, from equation (0)

$$z \quad -4 \cdot (0 + \delta) \quad -3 \cdot 0 \quad = \quad 0\,.$$

z value will increase from 0 to $4\delta = 8$. The value of z goes up at a rate of 4 per x_1. Point A is not optimal, and we can travel to either point B or point E. Arbitrarily select point E.

The Simplex Method

Realize that the travel from A to E was greatly facilitated by the "standard" algebraic configuration of the constraints:

1. Each of constraints (1)–(4) contains a basic variable that has a coefficient of 1 and appears in that equation only.

2. The constants on the rhs of equations (1)–(4) are all nonnegative.

3. The basic variables do not appear in constraint (0).

Going from A to E:

1. the status of x_2 from being nonbasic to being basic

2. the status of s_4 from being basic to nonbasic.

3. Convert the constraint set into the following "target" configuration,

The standard form for new basis:

$$\begin{array}{lllllllll} z & & +? & & & & +? & = & ? & (0) \\ & & +? & +s_1 & & & +? & = & ? & (1) \\ & & +? & & +s_2 & & +? & = & ? & (2) \\ & & +? & & & +s_3 & +? & = & ? & (3) \\ & x_1 & +? & & & & +? & = & ?\,, & (4) \end{array}$$

Use two operations: (i) multiplying an equation by a nonzero number (ii) adding one equation to another. These are called **row operations**.

Consider equation (4) first. x_1 has a coefficient of 2; how to make it 1?

Mquation (4) by 1/2:

$$x_1 \quad +(1/2)x_2 \qquad\qquad +(1/2)s_4 \quad = \quad 2\,.$$

The Simplex Method

Consider equation (0). Eliminate $-4x_1$.

Multiply the original equation (4) by 2 and add to the original equation (0).

$$z \quad -x_2 \qquad\qquad +2s_4 \;=\; 8\,.$$

Consider equation (1). Eliminate $2x_1$ from equation (1).

Multiply the original equation (4) by -1 and add the outcome to equation (1):

$$+2x_2 \quad +s_1 \qquad\qquad -s_4 \;=\; 2\,.$$

Consider equation (2). Eliminate $-3x_1$.

Multiply the original equation (4) by $3/2$ and add the outcome to equation (2):

$$+(7/2)x_2 \qquad +s_2 \qquad +(3/2)s_4 \;=\; 9\,.$$

x_1 does not appear in equation (3), no need to revise that equation.

The Simplex Method

We have arrived at the following (target) equation system:

$$
\begin{array}{llllllllll}
z & & -x_2 & & & & +2s_4 & = & 8 & (0) \\
 & & +2x_2 & +s_1 & & & -s_4 & = & 2 & (1) \\
 & & +(7/2)x_2 & & +s_2 & & +(3/2)s_4 & = & 9 & (2) \\
 & & +2x_2 & & & +s_3 & & = & 5 & (3) \\
 & x_1 & +(1/2)x_2 & & & & +(1/2)s_4 & = & 2\,. & (4)
\end{array}
$$

This is called **Gaussian elimination**.

Should I stay (at the current bfs) or should I go (to another bfs)? Increase the value of a nonbasic variable to go to an adjacent basic feasible solution.

x_2 and s_4 are nobasic and have coefficients -1 and $+2$. Objective value increases at a rate 2 with x_2 and at a rate -1 with s_4. The objective-function value can be improved by increasing the value of x_2, current bfs is not optimal.

Optimality test? **Optimality criterion**? **Entering variable**? Entering variable if there are multiple candidates? The standard Simplex method selects the nonbasic variable with the smallest coefficient for maximization (myopic!).

The Simplex Method

Entering x_2 into basis: (1) perform the ratio test to find the leaving variable (turns out to be s_4), (2) boost the value of x_2, (3) put the constraint set into the standard form :

$$
\begin{array}{ccccccccl}
z & & & +(1/2)s_1 & & & +(3/2)s_4 & = & 9 & (0) \\
& & +x_2 & +(1/2)s_1 & & & -(1/2)s_4 & = & 1 & (1) \\
& & & -(7/4)s_1 & +s_2 & & +(13/4)s_4 & = & 11/2 & (2) \\
& & & -s_1 & & +s_3 & +s_4 & = & 3 & (3) \\
& +x_1 & & -(1/4)s_1 & & & +(3/4)s_4 & = & 3/2\,. & (4)
\end{array}
$$

The nonbasic variables : s_1 and s_4. The basic variables : x_1, x_2, s_2, and s_3. The bfs is $(3/2,\ 1,\ 0,\ 11/2,\ 3,\ 0)$, which corresponds (as expected) to point D, the objective function value equals 9, taken from the rhs of equation (0).

Should I stay or should I go?

At last the independence (from graph) is won!

Remarks

1. It is often not necessary for the Simplex method to visit all of the basic feasible solutions before determining which one is optimal. In our example, there are five basic feasible solutions, but only three out of these five are (explicitly) visited.

The Simplex Method in Tabular Form

In its original **algebraic form**, our problem is:

Maximize z

Subject to:

$$
\begin{array}{rrrrrrrcl}
z & -4x_1 & -3x_2 & & & & & = & 0 \quad (0) \\
 & 2x_1 & +3x_2 & +s_1 & & & & = & 6 \quad (1) \\
 & -3x_1 & +2x_2 & & +s_2 & & & = & 3 \quad (2) \\
 & & 2x_2 & & & +s_3 & & = & 5 \quad (3) \\
 & 2x_1 & +x_2 & & & & +s_4 & = & 4 \quad (4)
\end{array}
$$

$$x_1,\ x_2,\ s_1,\ s_2,\ s_3,\ s_4 \geq 0\,.$$

Write down the coefficients of the variables and rhs's:

	z	x_1	x_2	s_1	s_2	s_3	s_4	RHS
R_0 :	1	-4	-3	0	0	0	0	0
R_1 :	0	2	3	1	0	0	0	6
R_2 :	0	-3	2	0	1	0	0	3
R_3 :	0	0	2	0	0	1	0	5
R_4 :	0	2	1	0	0	0	1	4

The Simplex Method in Tabular Form

Did we forget about the nonnegativity constraints?

Associated with this *initial* tableau, the nonbasic variables: x_1 and x_2 and the basic variables: s_1, s_2, s_3, and s_4. The initial bfs is: $(x_1, x_2, s_1, s_2, s_3, s_4) = (0, 0, 6, 3, 5, 4)$. with an objective value of 0.

Consider R_0. Since the coefficients of x_1 and x_2 are both negative, the current solution is not optimal. What would be the entering variable? **Pivot column** terminology borrowed from Gaussian Elimination.

x_1 has the most negative coefficient in R_0. Enter it into the basis.

There are other rules to find entering variable. Any variable with a negative coefficient in R_0 can be chosen in a maximization problem.

The Simplex Method in Tabular Form

Do a Ratio test to find maximum possible increase in x_1.

Basic Variable	z	$\mathbf{x_1}$	x_2	s_1	s_2	s_3	s_4	RHS	Ratio Test	
	1	**-4**	-3	0	0	0	0	0		
s_1	0	**2**	3	1	0	0	0	6	$6/2 = 3$	
s_2	0	**-3**	2	0	1	0	0	3	$-$	
s_3	0	**0**	2	0	0	1	0	5	$-$	
s_4	0	**2**	1	0	0	0	1	4	$4/2 = 2$	$\leftarrow$ Minimum

We did not compute a ratio for R_2 and R_3, why?

s_4 is leaving, we call R_4 the **pivot row**. **Pivot element**?

The Simplex Method in Tabular Form

The new basis will be x_1, s_1, s_2, and s_3. Need a new tableau in the configuration specified below.

Basic Variable	z	x_1	x_2	s_1	s_2	s_3	s_4	RHS
	1	0	?	0	0	0	?	?
s_1	0	0	?	1	0	0	?	?
s_2	0	0	?	0	1	0	?	?
s_3	0	0	?	0	0	1	?	?
x_1	0	1	?	0	0	0	?	?

To create this target tableau, we will employ row operations.

	z	x_1	x_2	s_1	s_2	s_3	s_4	RHS
$2 \cdot R_4 + R_0$:	1	0	-1	0	0	0	2	8
$(-1) \cdot R_4 + R_1$:	0	0	2	1	0	0	-1	2
$(3/2) \cdot R_4 + R_2$:	0	0	$7/2$	0	1	0	$3/2$	9
$0 \cdot R_4 + R_3$:	0	0	2	0	0	1	0	5
$(1/2) \cdot R_4$:	0	1	$1/2$	0	0	0	$1/2$	2

The Simplex Method in Tabular Form

The new basis: x_1, s_1, s_2, and s_3. The new bfs: $(x_1, x_2, s_1, s_2, s_3, s_4) = (2, 0, 2, 9, 5, 0)$. The new objective value 8. Should I stay or should I go?

x_2 is now the entering variable, the x_2-column is the new pivot column. To determine the pivot row, we again conduct a ratio test.

Basic Variable	z	x_1	$\mathbf{x_2}$	s_1	s_2	s_3	s_4	RHS	Ratio Test	
	1	0	**-1**	0	0	0	2	8		
s_1	0	0	**2**	1	0	0	-1	2	$2/2 = 1$	$\leftarrow$ Minimum
s_2	0	0	**7/2**	0	1	0	$3/2$	9	$9/(7/2) = 18/7$	
s_3	0	0	**2**	0	0	1	0	5	$5/2$	
x_1	0	1	**1/2**	0	0	0	$1/2$	2	$2/(1/2) = 4$	

This shows that the new pivot row will be R_1, and s_1, will be the leaving variable.

The Simplex Method in Tabular Form

With the entry 2 (Which?) as the pivot element, we now go through another set of row operations to obtain the new tableau below.

	z	x_1	x_2	s_1	s_2	s_3	s_4	RHS
$(1/2) \cdot R_1 + R_0$:	1	0	0	$1/2$	0	0	$3/2$	9
$(1/2) \cdot R_1$:	0	0	1	$1/2$	0	0	$-1/2$	1
$(-7/4) \cdot R_1 + R_2$:	0	0	0	$-7/4$	1	0	$13/4$	$11/2$
$(-1) \cdot R_1 + R_3$:	0	0	0	-1	0	1	1	3
$(-1/4) \cdot R_1 + R_4$:	0	1	0	$-1/4$	0	0	$3/4$	$3/2$

The bfs associated with this new tableau is $(3/2, 1, 0, 11/2, 3, 0)$, with a corresponding objective-function value of 9. Should I stay or should I go?

Remarks

1. What, if the entering variable column (printed in **boldface**) has no positive coefficient? No ratio test can be performed!