the closed-loop system. In the future, more effort is going to be made toward this direction. Also, the constraints on the agent's dynamics, such as the bounded input and the nonholonomic constraint, which are inevitable in practice, will be very interesting topics for the future research.

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Exponential H_∞ Synchronization of General Discrete-Time Chaotic Neural Networks With or Without Time Delays

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***Abstract*—This brief studies exponential H_∞ synchronization of a class of general discrete-time chaotic neural networks with external disturbance. On the basis of the drive–response concept and H_∞ control theory, and using Lyapunov–Krasovskii (or Lyapunov) functional, state feedback controllers are established to not only guarantee exponential stable synchronization between two general chaotic neural networks with or without time delays, but also reduce the effect of external disturbance on the synchronization error to a minimal H_∞ norm constraint. The proposed controllers can be obtained by solving the convex optimization problems represented by linear matrix inequalities. Most discrete-time chaotic systems with or without time delays, such as Hopfield neural networks, cellular neural networks, bidirectional associative memory networks, recurrent multilayer perceptrons, Cohen–Grossberg neural networks, Chua's circuits, etc., can be transformed into this general chaotic neural network to be H_∞ synchronization controller designed in a unified way. Finally, some illustrated examples with their simulations have been utilized to demonstrate the effectiveness of the proposed methods.**

***Index Terms*—Chaotic neural network, discrete-time system, drive-response conception, eigenvalue problem (EVP), H_∞ synchronization.**

I. INTRODUCTION

IN the past decades, there has been much attention on dynamic neural networks due to their promising potential applications such as pattern recognition, signal processing, associative memories, and optimization. Some important results have been reported in [25], [26], and [28]. Dynamic neural networks have been found to exhibit some complex and unpredictable behaviors including stable equilibria, periodic oscillations, bifurcation and chaotic attractors. As a special neural network, chaotic neural network has been successfully applied in combinational optimization [12], associative memory [14], secure communication [21], chemical biology [8], etc. Since the pioneering work by Pecora and Carroll [23], originally proposed the drive-response (master-slave) concept

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for achieving the synchronization of coupled chaotic systems, researchers have also proposed a variety of alternative schemes for ensuring the control and synchronization of chaotic neural networks with or without delays, which include linear and nonlinear feedback control, adaptive design control, impulsive control method, and invariant manifold method, among many others (see [3]–[6], [15], [18], [19], [29]–[31], and references cited therein).

In real physical systems, some noises or disturbances always exist that may cause instability and poor performance. Therefore, how to reduce the effect of the noises or disturbances in synchronization process for chaotic systems becomes an important issue. Suykens *et al.* [27] firstly adopted the H_∞ control concept to reduce the effect of the disturbance for chaotic synchronization problem of chaotic Lur'e systems. [9] and [13] investigated the H_∞ synchronization problem for a general class of chaotic systems with external disturbance via dynamic feedback approach. On the other hand, there has been increasing interest in time-delay chaotic systems since chaos phenomenon in time-delay systems first found by Mackey and Glass [20]. The H_∞ synchronization problem for time-delayed chaotic systems is also investigated by some researchers [10], [11], [22].

However, to our best knowledge, the above aforementioned methods and many other existing synchronization methods are only applied to the continuous-time chaotic systems. It seems not much (if any) study on the H_∞ synchronization for discrete-time delayed chaotic systems with external disturbance. It is well known that discrete-time systems play a very important role in digital signal analysis and processing. Especially, the discrete-time neural networks have been found intensive applications, such as bidirectional associative memory, nonlinear output regulation and adaptive tracking. Furthermore, the discrete-time chaotic neural networks are more suitable to realize time-series prediction, and secure communication, etc. Therefore, here we will combine the H_∞ control concept and Lyapunov stability theory to investigate the optimal H_∞ synchronization problem for a class of discrete-time chaotic neural networks with external disturbances. We are inspired by the standard neural network model in [16] and [17] and put forward this general discrete-time chaotic neural network, which is the interconnection of a linear dynamic system and a bounded static nonlinear operator. Most chaotic systems with or without time delays, such as Hopfield neural networks, cellular neural networks (CNNs), bidirectional associative memory (BAM) networks, recurrent multilayer perceptrons (RMLPs), Cohen–Grossberg neural networks (CGNNs), and Chua's circuits, etc., can be transformed into this general chaotic neural network to be H_∞ synchronization controller designed in a unified way. State feedback controller for the synchronization between two general discrete-time chaotic neural networks is proposed. By the state feedback control scheme, the closed-loop error system is exponentially stable and the H_∞-norm from the disturbance to controlled output is reduced to a lowest level.

Notation: $\Re^n$ denotes n dimensional Euclidean space and $\Re^{n\times m}$ is the set of all $n\times m$ real matrices. $l_2[0,\infty)$ is the space of square integrable vectors. I denotes the identity matrix of appropriate order. A^{T} means the transpose of the matrix A. diag $\{\cdots\}$ denotes the block diagonal matrix. $*$ denotes the symmetric parts. $\lambda_M(A)$ and $\lambda_m(A)$ denote the maximal and minimal eigenvalue of a square matrix A, respectively. $||x||$ denotes the Euclid norm of the vector x. Matrix inequalities $X > Y$ and $X \geq Y$ mean that $X - Y$ is positive definite and positive semi-definite, respectively. If $X \in \Re^p$ and $Y \in \Re^q$, $C(X;Y)$ denotes the space of all continuous mapping $\Re^p \to \Re^q$.

II. Problem Formulation

In this brief, we consider the following chaotic delayed neural network model [16], [17]:

$$\begin{cases} x(k+1) = Ax(k) + A_d x(k-\tau) + B_p\phi(\xi(k)) \\ \xi(k) = C_q x(k) + C_{qd} x(k-\tau) + D_p\phi(\xi(k)) \\ z_x(k) = Cx(k) \end{cases} \tag{1}$$

with the initial condition function $x(k) = \varpi(k)$, $\forall k \in [-\tau, 0]$, where $x(k) \in \Re^n$ is the state vector associated with the neurons, $A \in \Re^{n\times n}$, $A_d\in \Re^{n\times n}$, $B_p\in \Re^{n\times L}$, $C_q\in \Re^{L\times n}$, $C_{qd}\in \Re^{L\times n}$, $D_p\in \Re^{L\times L}$, and $C\in \Re^{l\times n}$ are the corresponding state-space matrices, $\phi \in C(\Re^L; \Re^L)$ is nonlinear activation function satisfying $\phi(0) = 0$, $\xi \in \Re^L$ is the input vector of ϕ, $\tau \geq 1$ is the transmission delay, $\varpi(\cdot)$ is the given continuous function on $[-\tau, 0]$, $z_x(k) \in \Re^l$ is the output vector, and $L\in\Re$ is the total number of neurons in the hidden layers and output layer of the neural network.

In this brief, we assume that the activation functions in (1) are monotonically nondecreasing and globally Lipschitz [9], [10], [13], [22]. That is, there exist positive scalar h_i such that

$$0 \leq [\phi_i(\mu) - \phi_i(\upsilon)]/(\mu - \upsilon) \leq h_i,\ i = 1, \ldots, L \tag{2}$$

for all arbitrary μ, $\upsilon \in \Re$.

Remark 11: The chaotic delayed neural network model (1) unifies not only several well-known discrete-time dynamic neural networks with or without delays such as Hopfield neural networks, CNNs, BAM networks, RMLPs, and CGNNs, etc., but also Chua's circuits and Lur'e systems. In Section IV, we will illustrate that these neural network models are special examples of (1). On the other hand, the system (1) reduces to a neural network without delays if $\tau = 0$ or $A_d = 0$ and $C_{qd} = 0$.

The synchronization problem of system (1) is considered using the drive-response configuration [23]. This is, if system (1) is regarded as the drive system, a suitable response system with control input should be constructed to synchronize the drive system. According to the above drive-response concept, unidirectionally coupled chaotic systems can be described by the following equations:

$$\begin{cases} y(k+1) = Ay(k) + A_d y(k-\tau) + B_p\phi(\zeta(k)) \\ \qquad\qquad + u(k) + Dw(k) \\ \zeta(k) = C_q y(k) + C_{qd} y(k-\tau) + D_p\phi(\zeta(k)) \\ z_y(k) = Cy(k) \end{cases} \tag{3}$$

with the initial condition function $y(k) = \sigma(k)$, $\forall k \in [-\tau, 0]$, where $y(k) \in \Re^n$ is the state vector of response system, $D \in \Re^{n\times s}$ is constant matrix, $\sigma(\cdot)$ is the given continuous function on $[-\tau, 0]$, $w(k) \in \Re^s$ is external disturbance which belongs to $l_2[0, \infty)$, $z_y(k) \in \Re^l$ is the output of the response system, and $u(k) \in \Re^n$ is a unidirectionally coupled term, which is regarded as the control input and will be appropriately designed such that the specific control objective is achieved.

Now, we define the synchronization error signal $e(k) = y(k) - x(k)$, where $x(k)$ and $y(k)$ are the state variables of drive system (1) and response system (3), respectively. Therefore, the error dynamical system between (1) and (3) is given as follows:

$$\begin{cases} e(k+1) = Ae(k) + A_d e(k-\tau) + B_p \psi(\eta(k)) \\ \qquad\qquad + u(k) + Dw(k) \\ \eta(k) = C_q e(k) + C_{qd} e(k-\tau) + D_p \psi(\eta(k)) \\ z_e(k) = Ce(k) \end{cases} \tag{4}$$

where $e(k) \in \Re^n$, $z_e(k) = z_y(k)\text{-}z_x(k)$, $\eta(k) = \zeta(k) - \xi(k)$, and $\psi(\eta(k)) = \phi(\zeta(k)) - \phi(\xi(k)) = \phi(\eta(k) + \xi(k)) - \phi(\xi(k))$, therefore $\psi(0) = 0$. Since all the $\phi_i(\cdot)$ are globally Lipschitz, $\psi_i(\cdot)$ satisfy the sector conditions, i.e., for each $i = 1, \ldots, L$

$$\begin{gathered} 0 \le \psi_i(\eta_i(k))/\eta_i(k) \le h_i \\ \text{or } \psi_i(\eta_i(k)) \cdot [\psi_i(\eta_i(k)) - h_i \eta_i k)] \le 0. \end{gathered} \tag{5}$$

Next, in order to synchronize between drive system (1) and response one (4) in the sense of H_∞ theory [1], let us consider the following state and time-delay state feedback controller:

$$u(k) = K_1 e(k) + K_2 e(k-\tau) \tag{6}$$

where $K_1 \in \Re^{n\times n}$ and $K_2 \in \Re^{n\times n}$ are feedback gains to be scheduled. With the control law (6), the error dynamics can be expressed by the following form:

$$\begin{cases} e(k+1) = \bar{A}e(k) + \bar{A}_d e(k-\tau) + B_p \psi(\eta(k)) + Dw(k) \\ \eta(k) = C_q e(k) + C_{qd} e(k-\tau) + D_p \psi(\eta(k)) \\ z_e(k) = Ce(k) \end{cases} \tag{7}$$

where $\bar{A} = A + K_1$, $\bar{A}_d = A_d + K_2$. Since $\psi(0) = 0$, (7) admits a trivial solution $e(k) \equiv 0$ in the absence of external disturbance $w(k)$.

Before stating the main results, we first need the following definition.

Definition 1 (Exponential H_∞ synchronization) [9]–[11], [13], [22]: The drive system (1) and the response system (3) are said to be exponentially synchronized in H_∞ sense if:

1) there exist constants $\lambda(\alpha) \ge 1$ and $\alpha > 0$ under $w(k) \equiv 0$ such that $\| x(k) - y(k) \| \le \lambda(\alpha) \sup_{-\tau \le i \le 0} \|y(i) - x(i)\| \exp(-\alpha k)$, for any $k \ge 0$. Moreover, the constant α is defined as the exponential synchronization rate;

2) the following condition holds under zero initial condition with a given positive constant γ:

$$J = \sum_{k=0}^{\infty} \exp(2\alpha k)[z_e^T(k) z_e(k) - \gamma^2 w^{\mathrm{T}}(k) w(k)] < 0$$
$$\left(\text{i.e.,} \sup_{w(k)\neq 0, w(k)\in l_2[0,\infty)} \frac{\|z_e(k)\|}{\|w(k)\|} < \gamma \right). \tag{8}$$

Then, the controller $u(k)$ is said to be the H_∞ synchronization controller with the disturbance attenuation γ. The parameter γ is called the H_∞-norm bound of the controller. If we find a minimal positive γ to satisfy the above conditions, the controller (6) is an optimal H_∞ synchronizer.

III. Main Results

Theorem 1: For given $\alpha > 0$, if there exist positive definite matrices R and Γ, diagonal semi-positive definite matrices Σ and Λ, a positive scalar γ, and nonzero matrices Y_1 and Y_2, that satisfy the following eigenvalue problem (EVP):

$$\text{minimize } \gamma^2 \tag{9}$$

subject to

$$\begin{bmatrix} -R & RA + Y_1 & RA_d + Y_2 & RB_p & RD \\ * & \Pi_1 & 0 & C_q^{\mathrm{T}} H\Sigma + C_q^{\mathrm{T}} \Lambda & 0 \\ * & * & -\exp(-2\alpha\tau)\Gamma & C_{qd}^{\mathrm{T}} H\Sigma + C_{qd}^{\mathrm{T}} \Lambda & 0 \\ * & * & * & \Pi_2 & 0 \\ * & * & * & * & -\gamma^2 I \end{bmatrix} < 0 \tag{10}$$

where $H = \text{diag}\{h_1, h_2, \ldots, h_L\}$, $\Pi_1 = -\exp(-2\alpha)R + \Gamma + C^{\mathrm{T}} C$, and $\Pi_2 = D^{\mathrm{T}} p H\Sigma + \Sigma H D_p - 2\Sigma + \Lambda D_p + D_{p^T} \Lambda$, then, the drive system (1) and the response system (3) can be synchronized with a prescribed exponential synchronization rate α, and H_∞-norm bound of the controller (6) does not exceed γ. Moreover, the feedback gains of optimal H_∞ controller (6) are obtained as $K_1 = R^{-1} Y_1$ and $K_2 = R^{-1} Y_2$.

Proof: Substituting $Y_1 = RK_1$ and $Y_2 = RK_2$ into (10), and using the well-known Schur complement [2], (10) is equivalent to

$$M = \begin{bmatrix} \bar{A}^{\mathrm{T}} R\bar{A} + \Pi_1 & \bar{A}^{\mathrm{T}} R\bar{A}_d & \bar{A}^{\mathrm{T}} RB_p + C_q^{\mathrm{T}}\Lambda + C_q^{\mathrm{T}} H\Sigma & \bar{A}^{\mathrm{T}} RD \\ * & \bar{A}_d^{\mathrm{T}} R\bar{A}_d - \exp(-2\alpha\tau)\Gamma & \bar{A}_d^{\mathrm{T}} RB_p + C_{qd}^{\mathrm{T}}\Lambda + C_{qd}^{\mathrm{T}} H\Sigma & \bar{A}_d^{\mathrm{T}} RD \\ * & * & B_p^{\mathrm{T}} RB_p + \Pi_2 & B_p^{\mathrm{T}} RD \\ * & * & * & D^{\mathrm{T}} RD - \gamma^2 I \end{bmatrix} < 0. \tag{11}$$

■

Firstly, we consider (7) with $w(k) = 0$; that is

$$\begin{cases} e(k+1) = \bar{A}e(k) + \bar{A}_d e(k-\tau) + B_p \psi(\eta(k)) \\ \eta(k) = C_q e(k) + C_{qd} e(k-\tau) + D_p \psi(\eta(k)). \end{cases} \tag{12}$$

For the error dynamical system (12), we define a positive definite Lyapunov–Krasovskii functional as

$$V(e(k),\eta(k)) = \exp(2\alpha k)e^{\mathrm{T}}(k)Pe(k) + \sum_{i=k-\tau}^{k-1}\exp(2\alpha i)e^{\mathrm{T}}(i)\Gamma e(i)$$
$$+2\exp[2\alpha(k-1)]\textstyle\sum_{j=1}^{L}\lambda_j\int_0^{\eta_j(k-1)}\psi_j(\sigma)d\sigma \qquad (13)$$

where $P = P^{\mathrm{T}} > 0$, $\Gamma = \Gamma^{\mathrm{T}} > 0$, $\alpha > 0$, and $\lambda_j \geq 0 (j = 1, 2, \ldots, L)$. Thus, $\forall e(k) \neq 0$, $\forall \eta(k) \neq 0$, $V(e(k), \eta(k)) > 0$, and $V(e(k), \eta(k)) = 0$ iff $e(k) = 0$ and $\eta(k) = 0$. We first give an estimation of the term of the integral $\int_0^{\eta_j(k)}\psi_j(\sigma)d\sigma$ by the sectors condition (5) and integral mean-value theorem. While $\eta_j(k) \geq 0$, we have

$$\int_0^{\eta_j(k)}\psi_j(\sigma)d\sigma = \eta_j(k)\psi_j(\beta) \leq \eta_j(k)\psi_j(\eta_j(k)) \qquad (14)$$

where $0 \leq \beta \leq \eta_j(k)$, $0 \leq \psi_j(\beta) \leq \psi_j(\eta_j(k))$. While $\eta_j(k) \leq 0$, (14) also holds, where $\eta_j(k) \leq \beta \leq 0$, $\psi_j(\eta_j(k)) \leq \psi_j(\beta) \leq 0$. From the sector condition (5), we have

$$2\exp(2\alpha k)\psi_i(\eta_i(k))\varepsilon_i[\psi_i(\eta_i(k)) - h_i\eta_i(k)] \leq 0 \qquad (15)$$

where $\varepsilon_i \geq 0 (i = 1, \ldots, L)$.

The difference of $V(e(k), \eta(k))$ along the solution to (12) is

$$\begin{aligned}\Delta V(e(k),\eta(k)) &= V(k+1) - V(k)\\ &\leq \exp(2\alpha k)\big\{\exp(2\alpha)[\bar{A}e(k) + \bar{A}_d e(k-\tau) + B_p\psi(\eta(k))]^{\mathrm{T}}\\ &\quad\times P[\bar{A}e(k) + \bar{A}_d e(k-\tau) + B_p\psi(\eta(k))] - e^{\mathrm{T}}(k)Pe(k)\\ &\quad+e^{\mathrm{T}}(k)\Gamma e(k) - \exp(-2\alpha\tau)e^{\mathrm{T}}(k-\tau)\Gamma e(k-\tau)\\ &\quad+2\sum_{j=1}^{L}\lambda_j\eta_j(k)\psi_j(\eta_j(k)) - 2\psi^{\mathrm{T}}(\eta(k))\Sigma\psi(\eta(k))\\ &\quad+2\psi^{\mathrm{T}}(\eta(k))H\Sigma\eta(k)\big\}\end{aligned} \qquad (16)$$

where $\Sigma = \mathrm{diag}\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_L\}$, and $\Sigma \geq 0$. Let $\exp(2\alpha)P = R$, we can get $P = \exp(-2\alpha)R$. We rewrite the above formulation of $\Delta V(e(k), \eta(k))$ as

$$\Delta V(e(k),\eta(k)) \leq \exp(2\alpha k)\begin{bmatrix} e(k)\\ e(k-\tau)\\ \psi(\eta(k))\end{bmatrix}^T G \begin{bmatrix} e(k)\\ e(k-\tau)\\ \psi(\eta(k))\end{bmatrix} \qquad (17)$$

where

$$G = \begin{bmatrix} \bar{A}^{\mathrm{T}}R\bar{A} - \exp(-2\alpha)R + \Gamma & \bar{A}^{\mathrm{T}}R\bar{A}_d & \bar{A}^{\mathrm{T}}RB_p + C_q^{\mathrm{T}}\Lambda + C_q^{\mathrm{T}}H\Sigma\\ * & \bar{A}_d^{\mathrm{T}}R\bar{A}_d - \exp(-2\alpha\tau)\Gamma & \bar{A}_d^{\mathrm{T}}RB_p + C_{qd}^{\mathrm{T}}\Lambda + C_{qd}^{\mathrm{T}}H\Sigma\\ * & * & B_p^{\mathrm{T}}RB_p + \Pi_2\end{bmatrix}.$$

Next, for (7) under zero initial condition, J in (8) is equivalent to

$$\begin{aligned}J &= \sum_{k=0}^{\infty}\exp(2\alpha k)[z_e^T(k)z_e(k) - \gamma^2 w^{\mathrm{T}}(k)w(k)]\\ &= \sum_{k=0}^{\infty}[\exp(2\alpha k)e^{\mathrm{T}}(k)C^{\mathrm{T}}Ce(k) - \exp(2\alpha k)\gamma^2 w^{\mathrm{T}}(k)w(k)\\ &\quad+\Delta V(e(k),\eta(k))] - V(\infty) + V(0)\\ &\leq \sum_{k=0}^{\infty}[\exp(2\alpha k)e^{\mathrm{T}}(k)C^{\mathrm{T}}Ce(k) - \exp(2\alpha k)\gamma^2 w^{\mathrm{T}}(k)w(k)\\ &\quad+\Delta V(e(k),\eta(k))]\\ &\leq \sum_{k=0}^{\infty}\exp(2\alpha k)\begin{bmatrix} e(k)\\ e(k-\tau)\\ \psi(\eta(k))\\ w(k)\end{bmatrix}^T M \begin{bmatrix} e(k)\\ e(k-\tau)\\ \psi(\eta(k))\\ w(k)\end{bmatrix}.\end{aligned} \qquad (18)$$

Since $M<0$ in (11), $J<0$ for any $[e^{\mathrm{T}}(k), e^{\mathrm{T}}(k-\tau), \psi^{\mathrm{T}}(\eta(k)), w^{\mathrm{T}}(k)]^{\mathrm{T}} \neq 0$ and $w \in l_2[0, \infty)$. By the Schur complement [2], $M < 0$ is equivalent to

$$\begin{bmatrix} \bar{A}^{\mathrm{T}}R\bar{A} - \exp(-2\alpha)R + \Gamma & \bar{A}^{\mathrm{T}}R\bar{A}_d & \bar{A}^{\mathrm{T}}RB_p + C_q^{\mathrm{T}}\Lambda + C_q^{\mathrm{T}}H\Sigma & \bar{A}^{\mathrm{T}}RD & C^{\mathrm{T}}\\ * & \bar{A}_d^{\mathrm{T}}R\bar{A}_d - \exp(-2\alpha\tau)\Gamma & \bar{A}_d^{\mathrm{T}}RB_p + C_{qd}^{\mathrm{T}}\Lambda + C_{qd}^{\mathrm{T}}H\Sigma & \bar{A}_d^{\mathrm{T}}RD & 0\\ * & * & B_p^{\mathrm{T}}RB_p + \Pi_2 & B_p^{\mathrm{T}}RD & 0\\ * & * & * & D^{\mathrm{T}}RD - \gamma^2 I & 0\\ * & * & * & * & -I\end{bmatrix} < 0. \qquad (19)$$

Since G is the principal minor of the left-hand side of (19), we have $G < 0$, that is, $\Delta V(k) \leq 0$ or $V(k) \leq V(0)$. However

$$\begin{aligned}V(0) &= e(0)^{\mathrm{T}}Pe(0) + \textstyle\sum_{i=-\tau}^{-1}\exp(2\alpha i)e^{\mathrm{T}}(i)\Gamma e(i)\\ &\quad+2\exp(-2\alpha)\textstyle\sum_{j=1}^{L}\lambda_j\int_0^{\eta_j(-1)}\psi_j(\sigma)d\sigma\\ &\leq \lambda_M(P)\|e(0)\|^2 + \lambda_M(\Gamma)\|\Theta\|^2\textstyle\sum_{i=-\tau}^{-1}\exp(2\alpha i)\\ &\quad+2\exp(-2\alpha)\lambda_M(H\Lambda)\|\eta(-1)\|^2\\ &\leq \Bigg[\lambda_M(P) + \lambda_M(\Gamma)\frac{\exp(-2\alpha) - \exp(-2\alpha(\tau+1))}{1 - \exp(-2\alpha)}\\ &\quad+2\exp(-2\alpha)\lambda_M(H\Lambda)\frac{\|\eta(-1)\|^2}{\|\Theta\|^2}\Bigg]\|\Theta\|^2\end{aligned} \qquad (20)$$

where $\|\Theta\| = \sup_{-\tau\leq i\leq 0}\|e(i)\|$, and

$$V(k) \geq \exp(2\alpha k)e^{\mathrm{T}}(k)Pe(k) \geq \exp(2\alpha k)\lambda_m(P)\|e(k)\|^2. \qquad (21)$$

Therefore, the convergence rates of $e(k)$ are

$$\|e(k)\| \leq \sqrt{\frac{\lambda_M(P)}{\lambda_m(P)} + \frac{\lambda_M(\Gamma)}{\lambda_m(P)}\cdot\frac{\exp(-2\alpha) - \exp(-2\alpha(\tau+1))}{1 - \exp(-2\alpha)} + \frac{\lambda_M(H\Lambda)}{\lambda_m(P)}\cdot\frac{2\exp(-2\alpha)\|\eta(-1)\|^2}{\|\Theta\|^2}}\,\|\Theta\|\exp(-\alpha k). \qquad (22)$$

From *Definition 1*, it concludes that the drive system (1) and the response system (3) are exponentially synchronized with an exponential synchronization rate α.

We hope that γ^2 is minimal such that (7) can reject the external disturbance as strong as possible. It requires solving the EVP (9), (10), which is a convex optimization problem and can be solved by using MATLABs linear matrix inequalities (LMI) Control Toolbox [7]. We thus complete the proof.

If $A_d = 0$ and $C_{qd} = 0$, (1) is a chaotic neural network without time delays, which is represented as

$$\begin{cases} x(k+1) = Ax(k) + B_p\phi(\xi(k)) \\ \xi(k) = C_q x(k) + D_p\phi(\xi(k)) \\ z_x(k) = Cx(k). \end{cases} \tag{23}$$

The response system corresponding to the drive system (23) is given by the following equations:

$$\begin{cases} y(k+1) = Ay(k) + B_p\phi(\zeta(k)) + u(k) + Dw(k) \\ \zeta(k) = C_q y(k) + D_p\phi(\zeta(k)) \\ z_y(k) = Cy(k). \end{cases} \tag{24}$$

The exponential synchronization controller is of the form

$$u(k) = Ke(k) \tag{25}$$

where $K \in \Re^{n\times n}$ is the state feedback gain. With the control law (25), the error dynamical system between (23) and (24) can be expressed by the following form:

$$\begin{cases} e(k+1) = (A+K)e(k) + B_p\psi(\eta(k)) + Dw(k) \\ \eta(k) = C_q e(k) + D_p\psi(\eta(k)) \\ z_e(k) = Ce(k). \end{cases} \tag{26}$$

Since $\psi(0) = 0$, (26) has a trivial solution $e(k) \equiv 0$. For the drive system (23) and the response system (24), we can use the following corollary to design optimal H_∞ synchronization controller (25).

Corollary 1: For given $\alpha > 0$, if there exist positive definite matrix R, diagonal semi-positive definite matrices Σ and Λ, a positive scalar γ, and nonzero matrix Y, that satisfy the following EVP:

$$\text{minimize}\ \gamma^2 \tag{27}$$

subject to

$$\begin{bmatrix} -R & RA+Y & RB_p & RD \\ * & -\exp(-2\alpha)R + C^{\mathrm{T}}C & C_q^{\mathrm{T}}H\Sigma + C_q^{\mathrm{T}}\Lambda & 0 \\ * & * & \Pi_2 & 0 \\ * & * & * & -\gamma^2 I \end{bmatrix} < 0 \tag{28}$$

then, the drive system (23) and the response system (24) can be synchronized with an prescribed exponential synchronization rate α, and H_∞-norm bound of the controller (25) does not exceed γ. Moreover, the feedback gain of optimal H_∞ controller (25) are obtained as $K = R^{-1}Y$.

The proof of Corollary 1 follows the same idea as that in the proof of Theorem 1, thus is omitted here. For Corollary 1, the following Lyapunov functional is chosen:

$$V(e(k), \eta_j(k)) = \exp(2\alpha k)e^{\mathrm{T}}(k)Pe(k) + 2\exp[2\alpha(k-1)]\sum_{j=1}^{L}\lambda_j\int_0^{\eta_j(k-1)}\psi_j(\sigma)d\sigma. \tag{29}$$

The H_∞ synchronization problem will be degenerated into the master-salve synchronization problem in the absence of the external disturbance $w(k)$. Similar to the proof of Theorem 1, we derive the following corollary to design exponential synchronization controller (6) for (1) and (3).

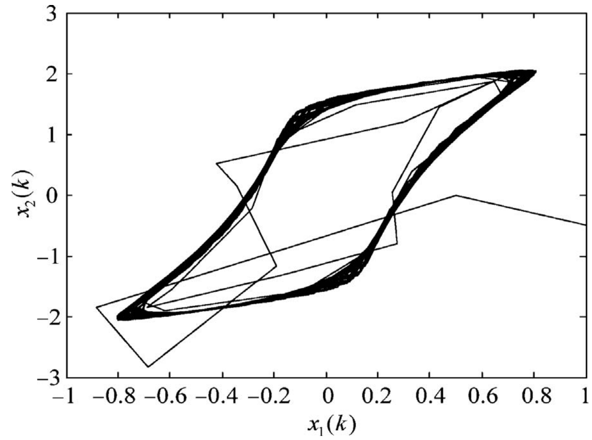

Fig. 1. Chaotic behavior of the delayed Hopfield neural network (31).

Corollary 2: For given $\alpha > 0$, if there exist positive definite matrix R, diagonal semi-positive definite matrices Σ and Λ, and nonzero matrix Y, that satisfy the following LMI:

$$\begin{bmatrix} -R & RA+Y_1 & RA_d+Y_2 & RB_p \\ * & \Pi_1 & 0 & C_q^{\mathrm{T}}H\Sigma + C_q^{\mathrm{T}}\Lambda \\ * & * & -\exp(-2\alpha\tau)\Gamma & C_{qd}^{\mathrm{T}}H\Sigma + C_{qd}^{\mathrm{T}}\Lambda \\ * & * & * & \Pi_2 \end{bmatrix} < 0 \tag{30}$$

then, the drive system (1) and the response system (3) can be synchronized with a prescribed exponential synchronization rate α. Moreover, the feedback gains of optimal H_∞ controller (6) are obtained as $K_1 = R^{-1}Y_1$ and $K_2 = R^{-1}Y_2$.

IV. Illustrative Examples

Example 1: Consider the following discrete-time delayed Hopfield neural network with two neurons:

$$\begin{cases} x_1(k+1) = 0.25x_1(k) - \tanh(x_1(k-1)) \\ \qquad\qquad + \tanh(0.5x_2(k-1)), \\ x_2(k+1) = x_2(k) - 4\tanh(0.5x_1(k-1)), \\ z_x(k) = 0.5x_1(k) + 0.5x_2(k). \end{cases} \tag{31}$$

Fig. 1 shows the chaotic behavior of (35) with the initial condition $[x_1(-1)\ x_2(-1)]^{\mathrm{T}} = [1\ -0.5]^{\mathrm{T}}$, $[x_1(0)\ x_2(0)]^{\mathrm{T}} = [0.5\ 0]^{\mathrm{T}}$. The response chaotic delayed Hopfield neural network with external disturbance is designed as follows:

$$\begin{cases} y_1(k+1) = 0.25y_1(k) - \tanh(y_1(k-1)) \\ \qquad\qquad + \tanh(0.5y_2(k-1)) + u_1(k) + 0.5w(k) \\ y_2(k+1) = y_2(k) - 4\tanh(0.5y_1(k-1)) \\ \qquad\qquad + u_2(k) + 0.5w(k) \\ z_y(k) = 0.5(y_1(k) + y_2(k)) \end{cases} \tag{32}$$

where external disturbance $w(k) \in l_2[0, \infty)$ is defined as

$$w(k) = -\frac{r}{0.1(k+1)} \tag{33}$$

where r is a random number taken from a uniform distribution over [0, 1]. We convert the delayed Hopfield neural

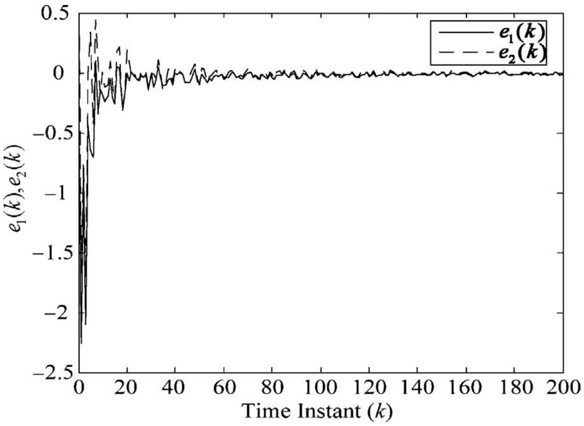

Fig. 2. Time responses of synchronization of discrete-time Hopfield neural networks without disturbance signal $w(k)$.

network (32) into (3), where $y(k) = [y_1(k)\ \ y_2(k)]^{\mathrm{T}}$, $A = \mathrm{diag}\{0.25,\ 1\}$, $A_d = 0_{2\times2}$, $B_p = \begin{bmatrix} 0 & -1 & 1 \\ -4 & 0 & 0 \end{bmatrix}$, $C_q = 0_{3\times2}$, $C_{qd} = \begin{bmatrix} 0.5 & 0 \\ 1 & 0 \\ 0 & 0.5 \end{bmatrix}$, $D_p = 0_{3\times3}$, $H = I_{3\times3}$, $\tau = 1$, $\phi_1(\zeta_1(k)) = tanh(0.5y_1(k-1))$, $\phi_2(\zeta_2(k)) = tanh(y_1(k-1))$, $\phi_3(\zeta_3(k)) = tanh(0.5y_2(k-1))$, $C = [0.5\,0.5]$, and $D = [0.5\,0.5]^T$. The controller (8) is employed to synchronize the delayed Hopfield neural networks (31) and (32). By using MATLABs LMI Control Toolbox [7], solving the EVP (9)–(10) given in Theorem 1, where the exponential synchronization rate $\alpha = 0.05$, we obtain the solutions of EVP and the controller parameters as

$$\gamma_{\min} = 5.2506\ R = \begin{bmatrix} 9.0517 & -0.0800 \\ -0.0800 & 2.5943 \end{bmatrix}$$

$$\Gamma = \begin{bmatrix} 7.7523 & -0.1684 \\ -0.1684 & 1.9702 \end{bmatrix}\ Y_1 = \begin{bmatrix} -2.2629 & 0.0800 \\ 0.0200 & -2.5943 \end{bmatrix}$$

$$Y_2 = \begin{bmatrix} 4.4462 & -2.2638 \\ 2.5550 & 0.0201 \end{bmatrix},$$

$$\Sigma = \mathrm{diag}\{22.2311, 8.0941, 13.2487\},$$

$$\Lambda = \mathrm{diag}\{0.0054, 0.0014, 0.0037\},$$

$$K_1 = \begin{bmatrix} -0.2500 & 0 \\ 0 & -1.0000 \end{bmatrix},\ K_2 = \begin{bmatrix} 0.5000 & -0.2501 \\ 1.0003 & 0 \end{bmatrix}.$$

In order to show control effort for synchronization between drive (31) and response system (32) in numerical simulation, the controller (6) with the above K_1 and K_2 is applied. First, without disturbance signal, the synchronization error between drive and response systems is given in Fig. 2. It shows that the envelopes of synchronization error waveform converge to zeros exponentially. To observe the H_∞ performance with disturbance attenuation, the response of the controlled output error $z_e(k)$ is depicted in Fig. 3. The time-delayed state feedback H_∞ controller (6) reduces the effect of the disturbance input $w(k)$ on the controlled output error $z_e(k)$ within a lowest level.

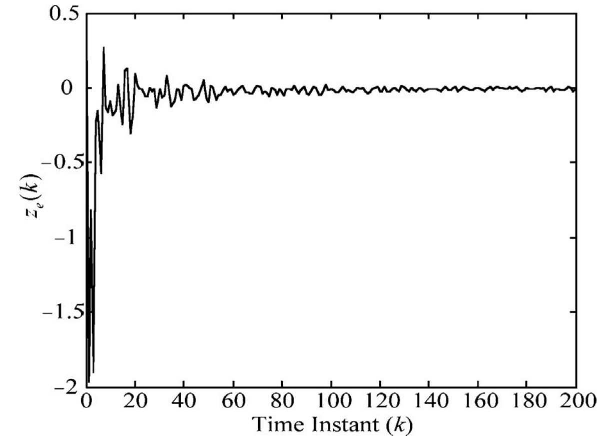

Fig. 3. Time response of the controlled output error $z_e(k)$ of discrete-time Hopfield neural networks with disturbance signal $w(k)$ defined as (33).

If the influence of external disturbance $w(k)$ is not taken into account, we can obtain the exponential synchronization controller by Corollary 2, where the exponential synchronization rate $\alpha = 0.1$. The parameters of LMI in (30) are

$$R = 10^3 \times \begin{bmatrix} 7.9825 & 0.0001 \\ 0.0001 & 1.9929 \end{bmatrix},\ \Gamma = 10^3 \times \begin{bmatrix} 6.5118 & -0.0001 \\ -0.0001 & 1.6162 \end{bmatrix}$$

$$Y_1 = 10^3 \times \begin{bmatrix} -1.9956 & -0.0001 \\ 0 & -1.9929 \end{bmatrix}$$

$$Y_2 = 10^3 \times \begin{bmatrix} 4.0037 & -2.0055 \\ 2.0027 & 0 \end{bmatrix}$$

$$\Sigma = 10^4 \times \mathrm{diag}\{1.6090, 0.6522, 1.0382\}$$

$$\Lambda = \mathrm{diag}\{78.9392\ , 20.0400, 51.4414\}$$

$$K_1 = \begin{bmatrix} -0.2500 & 0 \\ 0 & -1.0000 \end{bmatrix},\ K_2 = \begin{bmatrix} 0.5000 & -0.2512 \\ 1.0049 & 0 \end{bmatrix}.$$

There are few works on the synchronization of discrete-time chaotic neural network as far as we can refer. Euler method was used in [24] to a continuous-time chaotic neural network, resulting that a discrete-time drive system is obtained, and gave some theorems to ensure that the drive-response systems are globally impulsively exponentially synchronized. Comparing our methods with [24], we give the design procedure of synchronization controller with a prescribed exponential synchronization rate α, while [24] could not. On the other hand, in [24] the discretized continuous-time chaotic systems still have chaotic behavior based on the condition that the sample time is small enough. Here, chaotic and discrete-time systems are considered, and few constraints are need in the design procedure.

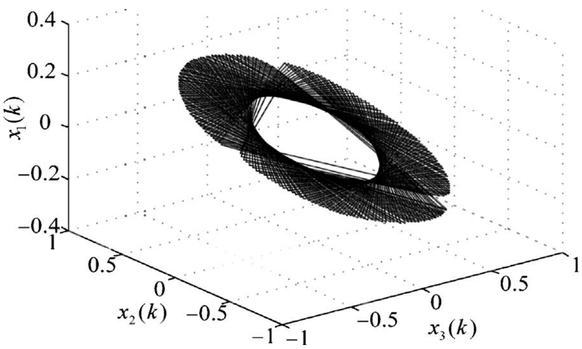

Fig. 4. Chaotic trajectory of system (34).

Example 2: We consider the following discrete-time chaotic RMLP without time delays

$$\begin{cases} x(k+1) = W\tanh(Vx(k)) \\ z_x(k) = Cx(k) \end{cases} \quad (34)$$

with the initial value $x_1(0) = 0$, $x_2(0) = -1$, and $x_3(0) = 0$, where

$$W = \begin{bmatrix} 0.9690 & 0.6967 & 0.2985 \\ -0.7473 & 3.2069 & 0.2840 \\ -2.7960 & 0.5360 & 0.9597 \end{bmatrix}$$

$$V = \begin{bmatrix} -0.9394 & -0.0078 & -0.5210 \\ -0.6861 & -0.1108 & -0.0729 \\ 0.0879 & 0.3845 & -0.7007 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}.$$

The behavior of the chaotic neural network is shown in Fig. 4.

The response chaotic RMLP is designed as follows:

$$\begin{cases} y(k+1) = W\tanh(Vy(k)) + u(k) + Dw(k) \\ z_y(k) = Cy(k) \end{cases} \quad (35)$$

where $D = [0.5\,1\,0.5]^{\mathrm{T}}$, $u(k) \in \Re^3$, $w(k) = \sin(2k)\exp(-0.03k)$. We transform the chaotic RMLP (35) into (24), where $A = 0_{3\times3}$, $B_p = W$, $C_q = V$, $D_p = 0_{3\times3}$, $H = I_{3\times3}$, $[\phi_1(\zeta_1(k))\ \phi_2(\zeta_2(k))\ \phi_3(\zeta_3(k))]^{\mathrm{T}} = \tanh(Vy(k))$. The controller (25) is employed to synchronize the RMLPs (34) and (35). According to Corollary 1, solving the EVP (27), (28), where the exponential synchronization rate $\alpha = 0.1$, we obtain the solutions of EVP and the feedback gain as

$$\gamma_{\mathrm{mim}} = 12.0809,\ \Sigma = \mathrm{diag}\{29.0179, 72.3497, 18.0385\}$$

$$\Lambda = \mathrm{diag}\{0.0058, 0.0050, 0.0041\}$$

$$R = \begin{bmatrix} 84.1708 & -21.6184 & 26.4712 \\ -21.6184 & 18.7611 & -11.8827 \\ 26.4712 & -11.8827 & 16.2237 \end{bmatrix}$$

$$X = \begin{bmatrix} 10.3895 & -8.2512 & 21.8593 \\ 13.0180 & 4.5503 & -3.4341 \\ -9.7349 & -4.5149 & 3.8209 \end{bmatrix}$$

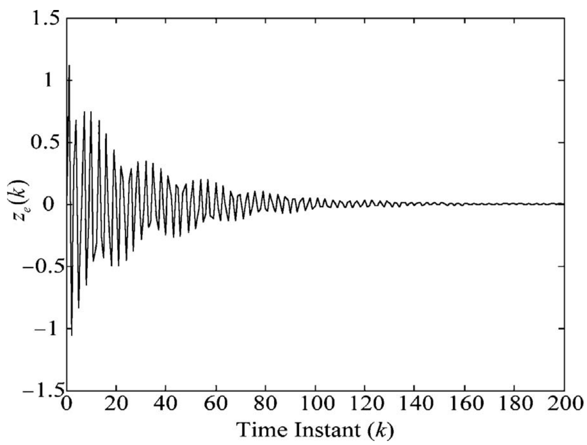

Fig. 5. Time response of the controlled output error $z_e(k)$ of discrete-time RMLPs with disturbance signal $w(k) = \sin(2k)\exp(-0.03k)$.

$$K = \begin{bmatrix} 0.6812 & -0.0150 & 0.3825 \\ 0.7364 & 0.1203 & 0.0216 \\ -1.1722 & -0.1656 & -0.3727 \end{bmatrix}.$$

When the state feedback law (25) with the above K is put on the response chaotic RMLP (35) with external disturbance $w(k)$, the response of the controlled output error $z_e(k)$ is shown in Fig. 5. It can be seen that the effect of the disturbance input $w(k)$ on the controlled output error $z_e(k)$ can be restricted in the lowest level.

From *Example 2*, we have notice that, although Su *et al.* [24] have provided a common chaotic neural network model to describe several well-known discrete-time chaotic neural networks (such as Hopfield neural network, cellular neural network, Chua's circuit, etc.), this model could not include RMLPs, and their approaches could not be used to solve the synchronization problem of chaotic RMLPs.

V. Conclusion

In this brief, we have provided a general discrete-time chaotic neural network model to unify several well-known dynamic neural networks with or without delays. Based on the Lyapunov stability theory and LMI approach, time-delay feedback controllers were designed to exponentially synchronize two general chaotic neural networks with different initial conditions and reduce the H_∞-norm from the disturbance input to the output error within a lowest level. By solving the EVPs, we have obtained the optimal H_∞-norm bound of the controller and the feedback gain matrices of optimal controllers in the response networks. Illustrative examples showed that most discrete-time chaotic neural networks with or without time delays can be converted into this general model (1), and exponential H_∞ synchronization controllers was designed by Theorem 1 and Corollary 1.

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Delay-Derivative-Dependent Stability for Delayed Neural Networks With Unbound Distributed Delay

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***Abstract*—In this brief, based on Lyapunov-Krasovskii functional approach and appropriate integral inequality, a new sufficient condition is derived to guarantee the global stability for delayed neural networks with unbounded distributed delay, in which the improved delay-partitioning technique and general convex combination are employed. The LMI-based criterion heavily depends on both the upper and lower bounds on time delay and its derivative, which is different from the existent ones and has wider application fields than some present results. Finally, three numerical examples can illustrate the efficiency of the new method based on the reduced conservatism which can be achieved by thinning the delay interval.**

***Index Terms*—Asymptotical stability, delayed neural networks (DNNs), LMI technique, Lyapunov-Krasovskii functional (LKF), unbounded distributed delay.**

I. Introduction

Recently, neural networks have been applied to various signal processing problems such as optimization, image processing, and associative memory design. In those applications, the key feature of the designed neural network is to be convergent. Meanwhile, since there inevitably exists communication delay which is the main source of oscillation and instability in biological and artificial neural network systems, great efforts have been made to analyze the dynamics of delayed neural networks (DNNs) and many elegant results have been reported (see [1]–[6], [8]–[13], [15], [16], [18]–[23], [25]–[28], and the references therein).

In practical applications, though it is difficult to describe the form of delay precisely, the bounds of delay and its derivative can be measured. Since Lyapunov functional approach imposes no restriction on delay's derivative and presents simple

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