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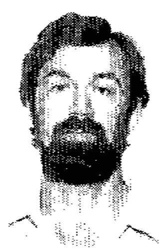

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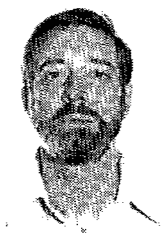

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The Graph Metric for Unstable Plants and Robustness Estimates for Feedback Stability

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***Abstract*—In this paper, a "graph metric" is defined that provides a measure of the distance between unstable multivariable plants. The graph metric induces a "graph topology" on unstable plants, which is the weakest possible topology in which feedback stability is robust. Using the graph metric, it is possible to derive estimates for the robustness of feedback stability *without* assuming that the perturbed and unperturbed plants have the same number of RHP poles. If the perturbed and unperturbed systems have the same RHP poles, then it is possible to obtain necessary and sufficient conditions for robustness with respect to a given class of perturbations. As an application of these results, the design of stabilizing controllers for unstable singularly perturbed systems is studied. Finally, the relationship of the graph metric to the "gap metric" introduced by Zames and El-Sakkary is studied in detail. In particular, it is shown that the robustness results of Zames and El-Sakkary do not enable one to conclude the *causality* of the perturbed system, whereas the present results do.**

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I. Introduction

THE objective of this paper is to study the robustness of feedback stability. Consider the standard feedback configuration shown in Fig. 1, where P represents the plant and C the compensator. Suppose this system is stable. Much research has been devoted to deriving conditions under which the system remains stable as P, C are replaced by some perturbed systems P_1, C_1 (see, e.g., the special issue [1]). Almost all of the research to date has been concentrated on the case where $P_1(C_1)$ has the same number of RHP poles as $P(C)$. Yet this is an artificial restriction that arises from the methods of analysis used. Suppose, for example, that $P=1/s$; then it is intuitively clear that the system will remain stable if P is replaced by $P_1=1/(s+\epsilon)$, provided ϵ is sufficiently small, and *it is immaterial whether ϵ is positive or negative*. The point is that in some sense both $1/(s-\epsilon)$ and $1/(s+\epsilon)$ are "close" to $1/s$, even though one system is stable and the other unstable.

Accordingly, the approach adopted in this paper is to define a notion of distance on the set of (possibly) unstable plants, and to obtain robustness margins for the feedback system in Fig. 1 in terms of the "distances" between P and P_1 and between C and

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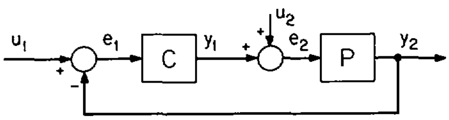

Fig. 1.

C_1. This would include, but would not be restricted to, the case where P_1 has the same number of RHP poles as P.

This paper is organized as follows. The rest of the introduction is devoted to demonstrating, by means of examples, some of the difficulties in defining a notion of proximity for unstable multivariable plants. In Section II we define a topology on the set of unstable plants, i.e., a notion of convergence of sequences. In this topology, a perturbation of the plant consists of perturbations of the "numerator" and "denominator" of the plant transfer matrix when it is expressed as a ratio of stable transfer matrices. We show that this is the weakest topology in which feedback stability is robust; that is, one can robustly stabilize against plant uncertainty if and only if the uncertainty can be expressed as perturbations in the stable numerator and denominator of the plant. In Section III we define a metric, i.e., a measure of distance, for unstable plants, and show that this metric induces the topology defined in Section II. Using this metric, we derive quantitative estimates for the robustness of feedback stability in Section IV. These results do *not* require the perturbed and unperturbed plants to have the same number of RHP poles, and as a consequence, they give only sufficient conditions for robustness. In case the perturbed and unperturbed plants have the same RHP poles, it is possible to obtain necessary and sufficient conditions for robustness, and this is done in Section V. In Section VI the results derived in the preceding sections are used to study the design of controllers for unstable singularly perturbed systems. The extension of the results given here to the case of distributed systems is discussed in Section VII.

In some earlier work [14], [15], a so-called "gap metric" is defined, and some robustness results are proved. In the Appendix we show that these robustness results do not enable one to conclude the *causality* of the perturbed system, whereas the present results do.

We now give some motivating discussion concerning the problem of defining a metric on the set of (possibly unstable) multivariable plants. While there are readily available and reasonable notions of distance on the set of *stable* plants, the issue is much more tricky in the case of *unstable* plants. The best way to illustrate this is by means of several examples.

As is customary, let $R(s)$ denote the set of rational functions in s with real coefficients. We use $\mathscr{S}$ (to suggest "stable") for the subset of $R(s)$ consisting of proper rational functions whose poles are all in the open left half-plane. Thus, $\mathscr{S}$ is the set of transfer functions of BIBO stable systems. For every f in $\mathscr{S}$, define

$$\|f\| = \sup_{\omega} |f(j\omega)| = \sup_{\mathrm{Re}\, s \geqslant 0} |f(s)|. \tag{1.1}$$

Then $\|\cdot\|$ defines a norm on $\mathscr{S}$, and the distance between two functions f and g in $\mathscr{S}$ is simply $\|f-g\|$. Thus, two BIBO stable systems are "close" if (and only if) their frequency responses are close at all frequencies. More generally, a sequence of functions $\{f_i\}$ in $\mathscr{S}$ converges to f in $\mathscr{S}$ if and only if $f_i(j\omega)$ converges to $f(j\omega)$ uniformly for all real ω (or equivalently, $f_i(s)$ converges to $f(s)$ uniformly for all s in the closed RHP). With this norm, $\mathscr{S}$ becomes a topological ring, i.e., addition and multiplication $\mathscr{S}$ are continuous. Thus, if $f_i \to f$ and $g_i \to g$, then $f_i + g_i \to f + g$ and $f_i g_i \to fg$.

The extension of the norm in (1.1) to stable multivariable plants is straightforward. Let $\mathrm{mat}(R(s))$ (resp. $\mathrm{mat}(\mathscr{S})$) denote the set of matrices with elements in $R(s)$ (resp. $\mathscr{S}$). For every $F \in \mathrm{mat}(\mathscr{S})$, define

$$\|F\| = \sup_{\omega} \bar{\sigma}(F(j\omega)) = \sup_{\omega} \left\{ \lambda_{\max}\left[F^{\dagger}(j\omega) F(j\omega) \right] \right\}^{1/2} \tag{1.2}$$

where $\bar{\sigma}$ denotes the largest singular value [2] and $\dagger$ denotes the conjugate transpose. This is a norm on $\mathrm{mat}(\mathscr{S})$ and this defines a metric on $\mathrm{mat}(\mathscr{S})$ in the obvious way. Moreover, a sequence of matrices $\{F^{(k)}\}$ in $\mathrm{mat}(\mathscr{S})$ converges to F in $\mathrm{mat}(\mathscr{S})$ [in the sense of the norm (1.2)] if and only if each of the component sequences $\{f_{ij}^{(k)}\}$ converges to f_{ij} [in the sense of the norm (1.1)].[1] Once again, addition and multiplication on $\mathrm{mat}(\mathscr{S})$ are continuous.

In the case of unstable plants, one can ask: what *should* proximity and convergence mean? Consider again the standard feedback configuration shown in Fig. 1, where P is a (possibly unstable) plant and C is the compensator. In other words, both P and C belong to $\mathrm{mat}(R(s))$. Let

$$H(P,C) = \begin{bmatrix} (I+PC)^{-1} & -P(I+CP)^{-1} \\ C(I+PC)^{-1} & (I+CP)^{-1} \end{bmatrix} \tag{1.3}$$

denote the transfer matrix relating (u_1, u_2) to (e_1, e_2).[2] The system in Fig. 1 is *stable* if $H(P,C) \in \mathrm{mat}(\mathscr{S})$, in which case we say that the pair (P,C) *is stable*, or that C *stabilizes* P. Now a reasonable notion of convergence in $\mathrm{mat}(R(s))$ is the following: A sequence of plants $\{P_i\}$ converges to P if there is a compensator C that stabilizes P as well as P_i for all large enough i, and in addition $H(P_i, C)$ converges to $H(P,C)$ in the sense of the norm (1.2). If one thinks of P as a nominal plant and of P_i as perturbations of P, then the above definition states that $P_i \to P$ if one can find a stabilizing compensator C for the nominal plant P that also stabilizes the perturbed plant P_i for large enough i, and in addition the perturbed *stable* closed-loop response $H(P_i, C)$ approaches the nominal *stable* closed-loop response $H(P,C)$. For reasons explained in Section II, we refer to the topology on $\mathrm{mat}(R(s))$ induced by the above notion of convergence as the *graph topology*.

This notion of convergence for unstable plants is very weak. There is no requirement that P_i and P should have the same McMillan degree, or that they should have the same number of RHP poles. The only requirement is that one can find a compensator C that stabilizes P as well as P_i, in such a way that $H(P_i, C)$ approaches $H(P,C)$. The contrapositive of this is that if P_i does *not* approach P, then either P_i and P cannot be simultaneously stabilized, or else the resulting closed-loop responses $H(P_i, C)$ and $H(P,C)$ will be widely different. Thus, the foregoing concept of convergence gives rise to the *weakest topology* in which feedback stability is a robust property, i.e., the weakest topology in which the function $P \to H(P,C)$ is continuous for some C. Note that we *do not* demand that *every* C that stabilizes P must also stabilize P_i for large enough i. As a result, in the case where P, P_i are all stable, the convergence of $\{P_i\}$ to P in the sense of the norm (1.2) implies that $P_i \to P$ in the graph topology as well—just take $C = 0$. (The converse is also true; see Lemma 2.2.)

To illustrate the above notion of convergence, consider the scalar case and let

$$P(s) = \frac{1}{(s+1)}. \tag{1.4}$$

[1]Mathematically, this means that the topology on $\mathrm{mat}(\mathscr{S})$ is the product topology obtained from $\mathscr{S}$.

[2]It is assumed that the system is well-posed so that the indicated inverses exist.

Let $\{\epsilon_i\}$ be any sequence, converging to zero, and let

$$P_i(s)=\frac{s-\epsilon_i}{(s+1)(s+\epsilon_i)}. \tag{1.5}$$

With a bit of "hand-waving," the reader can convince himself that, in order for C to stabilize both P as well as P_i for all large enough i, $C(0)$ must equal 0; but in this case, $H(P_i,C)|_{s=0}$ will not approach $H(P,C)|_{s=0}$. Thus, the family of plants in (1.5) does not represent a "valid perturbation" of P. As another example, suppose

$$P(s)=\frac{1}{s(s+1)},\qquad P_i(s)=\frac{s-\epsilon_i}{s(s+1)(s+\epsilon_i)}. \tag{1.6}$$

Then *no* compensator can be found that stabilizes P as well as all P_i for large enough i.

In the above examples, the nonconvergence of $\{P_i\}$ to P can be explained by the "illegal" RHP pole-zero cancellation at $s=0$ as $i\to\infty$. But this "intuition" can lead one astray in the case of multivariable systems. Consider

$$P(s)=\begin{bmatrix} 0 & 1 \\ \dfrac{2s^2-1}{s^2-1} & \dfrac{-s^2+s+1}{s^2-1}\end{bmatrix} \tag{1.7}$$

$$P_\epsilon(s)=\begin{bmatrix} \dfrac{-\epsilon s}{s-1} & \dfrac{s-1-\epsilon}{s-1} \\ \dfrac{2s^2-1}{s^2-1} & \dfrac{-s^2+s+1}{s^2-1}\end{bmatrix}. \tag{1.8}$$

Then as $\epsilon\to 0$ there is a pole-zero cancellation at $s=1$ in the $(1,2)$ component of P_ϵ. Nevertheless, P_ϵ can be shown to converge to P as $\epsilon\to 0$. In fact, the following stronger statement is true. Let C be *any* compensator that stabilizes P; then C also stabilizes P_ϵ for small enough ϵ, and in addition $H(P_\epsilon,C)\to H(P,C)$ as $\epsilon\to 0$.

As a final example, consider

$$P_\epsilon(s)=\begin{bmatrix}\dfrac{s+1}{s-1+\epsilon} & \dfrac{s+1}{s-1-\epsilon}\end{bmatrix} \tag{1.9}$$

$$P(s)=\begin{bmatrix}\dfrac{s+1}{s-1} & \dfrac{s+1}{s-1}\end{bmatrix}. \tag{1.10}$$

Then each component of P_ϵ converges to the corresponding component of P as $\epsilon\to 0$. Nevertheless, P_ϵ does *not* converge to P as $\epsilon\to 0$.

The preceding examples illustrate the difficulties involved in determining what is and what is not a "valid" perturbation of an unstable multivariable plant. In the next two sections we define a topology and a metric, respectively, that can be used to determine unambiguously whether or not a sequence $\{P_i\}$ converges to a limit candidate P.

II. The Graph Topology

In this section we define a topology on the set $\mathrm{mat}(R(s))$ of (possibly) unstable plants, and study some of its properties. The most important of these is that the topology presented here is the weakest one on $\mathrm{mat}(R(s))$ in which feedback stability is a robust property.

Recall [3] that every matrix in $\mathrm{mat}(R(s))$ has both a right-coprime factorization (RCF) as well as a left-coprime factorization (LCF) over the ring $\mathscr{S}$ of proper stable rational functions. Thus, if $P\in\mathrm{mat}(R(s))$, then there exist N, D, $\tilde{N}$, $\tilde{D}$, X, Y, $\tilde{X}$, $\tilde{Y}\in\mathrm{mat}(\mathscr{S})$ such that

$$P(s)=N(s)[D(s)]^{-1}=[\tilde{D}(s)]^{-1}\tilde{N}(s) \tag{2.1}$$

$$X(s)N(s)+Y(s)D(s)=I \tag{2.2}$$

$$\tilde{N}(s)\tilde{X}(s)+\tilde{D}(s)\tilde{Y}(s)=I,\qquad \text{for all } s. \tag{2.3}$$

An easy way to find an RCF and an LCF is the following [3]. Given $P\in\mathrm{mat}(R(s))$, define $P_1\in\mathrm{mat}(R(\lambda))$ by $P_1(\lambda)=P((1-\lambda)/\lambda)$, and then find an RCF and an LCF of P_1 over the ring of *polynomials* $R[\lambda]$ using standard methods [4], [5]. If $(N_1(\lambda),D_1(\lambda))$ is an RCF of $P_1(\lambda)$ over $R[\lambda]$, then $N(s)=N_1(1/(s+1))$, $D(s)=D_1(1/(s+1))$ gives an RCF of P over the ring $\mathscr{S}$. Similar remarks apply to LCF's.[3]

In [3] it is also shown that an RCF of an unstable plant parametrizes the graph of the plant in a simple way. Suppose $P\in\mathrm{mat}(R(s))$ is of order $n\times m$. Then the *graph* of P, denoted by $\mathscr{G}(P)$, consists of the subspace of L_2^{n+m} defined by[4]

$$\mathscr{G}(P)=\{(u,y)\in L_2^{n+m}: y=Pu\}. \tag{2.4}$$

Thus, the graph of P simply consists of the bounded (in the L_2-sense) input–output pairs corresponding to the possibly unstable plant P. If in particular $P\in\mathrm{mat}(\mathscr{S})$, then P maps *every* $u\in L_2^m$ into L_2^n, and

$$\mathscr{G}(P)=\{(u,Pu): u\in L_2^m\}. \tag{2.5}$$

If P does not belong to $\mathrm{mat}(\mathscr{S})$, then not every u in L_2^m gets mapped into L_2^n by P. In this case [3, Theorem 2]

$$\mathscr{G}(P)=\{(Dz,Nz): z\in L_2^m\} \tag{2.6}$$

where (N,D) is any RCF over $\mathscr{S}$ of P.[5] Note that if $P\in\mathrm{mat}(\mathscr{S})$, then (P,I) is an RCF of P, so that (2.6) reduces to (2.5).

With this background, we are ready to define a notion of convergence in $\mathrm{mat}(R(s))$.

Definition 2.1: A sequence of plants $\{P_i\}$ in $\mathrm{mat}(R(s))$ converges to P in the *graph topology* if there exist RCF's (N_i,D_i) of P_i and (N,D) of P such that $N_i\to N$, $D_i\to D$ in $\mathrm{mat}(\mathscr{S})$.[6]

Thus, $P_i\to P$ if one can factorize P_i and P in such a way that the right numerator and denominator of P_i, respectively, approach the right numerator and denominator of P. In this way, convergence of unstable plant sequences can be examined in terms of the convergence of stable plant sequences, which is described in Section I. It is immediate from (2.6) that, if $P_i\to P$ in the sense of Definition 2.1, then the *graph* of P_i converges to the *graph* of P (in a sense to be made precise in Section III). This is why the topology on $\mathrm{mat}(R(s))$ introduced in Definition 2.1 is referred to as the graph topology.

At this stage one can ask: i) does it matter which RCF of P is used, and ii) can LCF's be used instead of RCF's? These questions are answered in the following result from [7, Lemma 4.7].

Lemma 2.1: Suppose $\{P_i\}$ is a sequence in $\mathrm{mat}(R(s))$ and that $P\in\mathrm{mat}(R(s))$. Then the following are equivalent.

[3] From a numerical viewpoint, finding factorizations over $\mathscr{S}$ is a more stable operation than finding polynomial factorizations. This is because the set of units in $\mathscr{S}$ is open, whereas the set of units in $R[s]$ is not. This issue will be discussed in more detail elsewhere.

[4] Here P is used to denote a rational matrix in s as well as the corresponding operator mapping an input space of functions into an output space.

[5] Hereafter all factorizations are over $\mathscr{S}$ unless stated otherwise.

[6] Actually, in order to fully characterize the topology on $\mathrm{mat}(R(s))$, Definition 2.1 would have to be broadened to discuss the convergence of *nets* rather than sequences [6]. The reader who is familiar with point set topology can easily do this. In Section III it is shown that the graph topology of Definition 2.1 is actually induced by a *graph metric*. As a result, $\mathrm{mat}(R(s))$ is a first-countable topological space, and the graph topology is fully characterized by giving conditions for the convergence of *sequences*.

i) P_i converges to P in the graph topology.

ii) there exist RCF's (N_i, D_i) of P_i and (N, D) of P such that $N_i \to N$, $D_i \to D$ in $\mathrm{mat}(\mathscr{S})$.

iii) For *every* RCF (N, D) of P, there is a corresponding sequence $\{(N_i, D_i)\}$ of RCF's of P_i such that $N_i \to N$, $D_i \to D$ in $\mathrm{mat}(\mathscr{S})$.

iv) There exist LCF's $(\tilde{D}_i, \tilde{N}_i)$ of P_i and $(\tilde{D}, \tilde{N})$ of P such that $\tilde{D}_i \to \tilde{D}$, $\tilde{N}_i \to \tilde{N}$ in $\mathrm{mat}(\mathscr{S})$.

v) For *every* LCF $(\tilde{D}, \tilde{N})$ of P, there is a corresponding sequence $\{(\tilde{D}_i, \tilde{N}_i)\}$ of LCF's of P_i such that $\tilde{D}_i \to \tilde{D}$, $\tilde{N}_i \to \tilde{N}$ in $\mathrm{mat}(\mathscr{S})$.

Note that the equivalence of i) and ii) is just a restatement of Definition 2.1. Condition iii) states that it does not matter which RCF of P is used. Conditions iv) and v) are the "left" analogs of ii) and iii).

Now we come to the main results of this section. To make the theorem statements more compact, we use the following notation. If $P \in \mathrm{mat}(R(s))$, then $S(P)$ denotes the set of compensators that stabilize P. In other words,

$$S(P) = \{C \in \mathrm{mat}(R(s)) : H(P, C) \in \mathrm{mat}(\mathscr{S})\} \tag{2.7}$$

where $H(P, C)$ is defined in (1.3).

Theorem 2.1: Suppose $\{P_i\}$ converges to P in the graph topology. Then every C in $S(P)$ also belongs to $S(P_i)$ for large enough i; moreover, for every $C \in S(P)$, $H(P_i, C) \to H(P, C)$ in $\mathrm{mat}(\mathscr{S})$. Conversely, suppose $\{P_i\}$ is a sequence in $\mathrm{mat}(R(s))$, that $P \in \mathrm{mat}(R(s))$, and suppose there exists a $C \in S(P)$ such that $C \in S(P_i)$ for large enough i and $H(P_i, C) \to H(P, C)$ in $\mathrm{mat}(\mathscr{S})$; then $\{P_i\}$ converges to P in the graph topology.

Corollary 2.1.1: Suppose $\{P_i\}$ is a sequence in $\mathrm{mat}(R(s))$ and that $P \in \mathrm{mat}(R(s))$. Suppose there exists *one* $C \in S(P)$ such that $C \in S(P_i)$ for large enough i and $H(P_i, C) \to H(P, C)$ in $\mathrm{mat}(\mathscr{S})$. Then *every* compensator in $S(P)$ has these properties.

Theorem 2.1 shows that the graph topology is the weakest one on $\mathrm{mat}(R(s))$ in which the function $P \to H(P, C)$ is continuous for some C, i.e., the weakest one in which feedback stability is robust against plant perturbations. A related result, given in [7, Theorem 4.1], shows that the graph topology is the weakest one in which the function $(P, C) \to H(P, C)$ is continuous, i.e., the weakest one in which feedback stability is robust against *simultaneous* plant and compensator perturbations. The present theorem statement is much more direct, and its proof much less technical, than in [7]. The result corresponding to the one in [7] is given next.

Theorem 2.2: Suppose $\{P_i\}$, $\{C_i\}$ are sequences in $\mathrm{mat}(R(s))$, that $P, C \in \mathrm{mat}(R(s))$, and that (P, C) is stable. Suppose $P_i \to P$, $C_i \to C$ in the graph topology; then (P_i, C_i) is stable for large enough i and $H(P_i, C_i) \to H(P, C)$ in $\mathrm{mat}(\mathscr{S})$. Conversely, suppose (P_i, C_i) is stable and $H(P_i, C_i) \to H(P, C)$ in $\mathrm{mat}(\mathscr{S})$; then $P_i \to P$, $C_i \to C$ in the graph topology.

The proof of Theorem 2.1 is not very difficult, and is based on a few well-known facts that are summarized below [7], [8]: A function $u \in \mathscr{S}$ is called a *unit* of $\mathscr{S}$ if $1/u$ also belongs to $\mathscr{S}$. Clearly, u is a unit if and only if it has neither poles nor zeros in the closed RHP, and does not vanish at infinity (such functions are called *miniphase* in [9]). Let $\mathscr{U}$ denote the set of units of $\mathscr{S}$. Then $\mathscr{U}$ is an *open* subset of $\mathscr{S}$; in other words, if u is a unit, and if $\{u_i\}$ is a sequence in $\mathscr{S}$ converging to u, then u_i is also a unit for large enough i. Moreover, the map $u \to u^{-1}$ from $\mathscr{U}$ into itself is continuous; thus if u is a unit and $\{u_i\}$ is a sequence in $\mathscr{S}$ converging to u, then $\{u_i^{-1}\}$ converges to u^{-1}. Similar results hold in the multivariable case as well. A matrix $U \in \mathrm{mat}(\mathscr{S})$ is *unimodular* if its determinant is a unit of $\mathscr{S}$; in this case U^{-1} also belongs to $\mathrm{mat}(\mathscr{S})$. The set of unimodular matrices is open and inversion is continuous; thus if U is unimodular and $\{U_i\}$ is a sequence in $\mathrm{mat}(\mathscr{S})$ converging to U, then U_i is also unimodular for large enough i and $\{U_i^{-1}\}$ converges to U^{-1}.

Proof of Theorem 2.1: To prove the first part of the theorem, suppose $\{P_i\}$ converges to P in the graph topology, and select RCF's (N_i, D_i) of P_i and (N, D) of P such that $N_i \to N$, $D_i \to D$ in $\mathrm{mat}(\mathscr{S})$. Let C be *any* stabilizing compensator for P. Then from [7], [10] there exists an LCF $(\tilde{D}_c, \tilde{N}_c)$ of C such that $\tilde{D}_c D + \tilde{N}_c N = I$. Moreover,

$$H(P, C) = \begin{bmatrix} I - N\tilde{N}_c & -N\tilde{D}_c \\ D\tilde{N}_c & D\tilde{D}_c \end{bmatrix}. \tag{2.8}$$

Now define $\Delta_i = \tilde{D}_c D_i + \tilde{N}_c N_i$. Then, as $\Delta_i \to I$, it follows that Δ_i is unimodular for large enough i and that $\Delta_i^{-1} \to I$. Hence, C stabilizes P_i for large enough i. Moreover, since

$$H(P_i, C) = \begin{bmatrix} I - N_i\Delta_i^{-1}\tilde{N}_c & -N_i\Delta_i^{-1}\tilde{D}_c \\ D_i\Delta_i^{-1}\tilde{N}_c & D_i\Delta_i^{-1}\tilde{D}_c \end{bmatrix} \tag{2.9}$$

from (1.3), and since addition and multiplication are continuous on $\mathrm{mat}(\mathscr{S})$, it follows that $H(P_i, C) \to H(P, C)$ in $\mathrm{mat}(\mathscr{S})$.

To prove the second part of the theorem, suppose $\{P_i\}$ is a sequence in $\mathrm{mat}(R(s))$, that $P \in \mathrm{mat}(R(s))$, and suppose there is a $C \in S(P)$ such that $C \in S(P_i)$ for large enough i and $H(P_i, C) \to H(P, C)$. Let (N, D) be any RCF of P. Since C stabilizes P, there is an LCF $(\tilde{D}_c, \tilde{N}_c)$ of C such that $\tilde{D}_c D + \tilde{N}_c N = I$. Moreover, $H(P, C)$ is given by (2.8). By discarding a finite number of terms from the sequence $\{P_i\}$ we can assume that C stabilizes P_i for all i. This implies that there is an RCF (N_i, D_i) of P_i such that $\tilde{D}_c D_i + \tilde{N}_c N_i = I$. Further,

$$H(P_i, C) = \begin{bmatrix} I - N_i\tilde{N}_c & -N_i\tilde{D}_c \\ D_i\tilde{N}_c & D_i\tilde{D}_c \end{bmatrix} \tag{2.10}$$

from (1.3). As $H(P_i, C) \to H(P, C)$ in $\mathrm{mat}(\mathscr{S})$, (2.8) and (2.10) imply that

$$\begin{bmatrix} N_i\tilde{N}_c & N_i\tilde{D}_c \\ D_i\tilde{N}_c & D_i\tilde{D}_c \end{bmatrix} \to \begin{bmatrix} N\tilde{N}_c & N\tilde{D}_c \\ D\tilde{N}_c & D\tilde{D}_c \end{bmatrix} \quad \text{in } \mathrm{mat}(\mathscr{S}). \tag{2.11}$$

Multiplying both sides of (2.11) by $\begin{bmatrix} N \\ D \end{bmatrix}$ and noting that $\tilde{N}_c N + \tilde{D}_c D = I$ gives

$$\begin{bmatrix} N_i \\ D_i \end{bmatrix} \to \begin{bmatrix} N \\ D \end{bmatrix} \quad \text{in } \mathrm{mat}(\mathscr{S}). \tag{2.12}$$

Hence, $\{P_i\}$ converges to P in the graph topology. □

The proof of Theorem 2.2 is deferred to Section IV, as it makes use of some further concepts that are developed therein.

The graph topology gives a notion of convergence in $\mathrm{mat}(R(s))$. As $\mathrm{mat}(\mathscr{S})$ is a subset of $\mathrm{mat}(R(s))$, the graph topology also gives a notion of convergence on $\mathrm{mat}(\mathscr{S})$. We now show that this is the same as convergence in the sense of the norm (1.2). This shows that the graph topology on $\mathrm{mat}(R(s))$ is a genuine extension to unstable plants of the familiar topology for stable plants.

Lemma 2.2: Suppose $\{P_i\}$ is a sequence in $\mathrm{mat}(\mathscr{S})$, $P \in \mathrm{mat}(\mathscr{S})$, and $\|P_i - P\| \to 0$; then $P_i \to P$ in the graph topology. Conversely, suppose $\{P_i\}$ is a sequence in $\mathrm{mat}(R(s))$, $P \in \mathrm{mat}(\mathscr{S})$, and $P_i \to P$ in the graph topology; then $P_i \in \mathrm{mat}(\mathscr{S})$ for large enough i, and $\|P_i - P\| \to 0$.

Proof: To prove the first part of the lemma, note that $C = 0$ stabilizes P as well as all P_i, since $P, P_i \in \mathrm{mat}(\mathscr{S})$. Further, since $\|P_i - P\| \to 0$, it follows from (1.3) that $H(P_i, C) \to H(P, C)$ in $\mathrm{mat}(\mathscr{S})$. Hence, by Theorem 2.1, $P_i \to P$ in the graph topology.

To prove the second part of the lemma, note that (P, I) is an RCF of P, since $P \in \mathrm{mat}(\mathscr{S})$. By assumption, there exist RCF's (N_i, D_i) of P_i such that $N_i \to P$, $D_i \to I$. Since the set of unimodular matrices is open and inversion is continuous, this implies that D_i is unimodular for large enough i and that $D_i^{-1} \to I$ in $\mathrm{mat}(\mathscr{S})$. Hence, $P_i = N_i D_i^{-1}$ is in $\mathrm{mat}(\mathscr{S})$ for large enough i, and approaches P in $\mathrm{mat}(\mathscr{S})$, i.e., $\|P_i - P\| \to 0$. □

The second part of Lemma 2.2 shows that, in the graph topology, $\mathrm{mat}(\mathscr{S})$ is an open subset of $\mathrm{mat}(R(s))$. In words, this means that the set of all stable (unstable) plants is an open (closed) subset of the set of all plants.

We now develop some further details concerning the convergence of sequences in the graph topology. These results are often helpful in concluding nonconvergence.

Lemma 2.3: Suppose $\{P_i\}$ converges to P in the graph topology. Let $s_1,\cdots,s_l$ denote the poles of P in the closed RHP. Then we have the following.

i) Let $\mathscr{R}$ be any compact subset of the open RHP such that none of the s_i lies on the boundary of $\mathscr{R}$, and let ν denote the number of poles of P inside $\mathscr{R}$, counted according to their McMillan degrees. Then for large enough i, each P_i has exactly ν poles inside $\mathscr{R}$.

ii) Let $\mathscr{R}$ be any closed subset of the closed RHP that does not contain $s_1,\cdots,s_l$; if P has a pole at infinity, suppose in addition that $\mathscr{R}$ is bounded. Then $P_i(s)\to P(s)$ as $i\to\infty$, uniformly for all s in $\mathscr{R}$.

Remarks: In general, the number of *closed* RHP poles of P_i and of P need not be equal; for example, consider $P_i(s)=1/(s+\epsilon_i)$, $P(s)=1/s$, where $\{\epsilon_i\}$ is any sequence of positive numbers converging to zero.

Proof: Since $P_i\to P$ in the graph topology, there exist RCF's (N_i,D_i) of P_i and (N,D) of P such that $N_i\to N$, $D_i\to D$ in $\mathrm{mat}(\mathscr{S})$. To prove i), observe that $\det D_i\to\det D$ in $\mathscr{S}$. That is, $\det D_i(s)\to\det D(s)$ uniformly over the closed RHP. Recall [20, Appendix I] that the RHP poles of P are precisely the RHP zeros of $\det D$, and the McMillan degree of an RHP pole of P is equal to its multiplicity as a zero of $\det D$; similar remarks apply to P_i and $\det D_i$. Since $\det D$ and $\det D_i$ are both analytic in the open RHP, and since $\det D$ has no zeros on the boundary of $\mathscr{R}$, i) follows from applying the principle of argument to $\det D$ and $\det D_i$. To prove ii), observe that since $D_i(s)\to D(s)$ uniformly over the closed RHP, it follows that $[D_i(s)]^{-1}\to[D(s)]^{-1}$ wherever the latter is well-defined, i.e., wherever $\det D(s)$ is nonzero. Moreover, if we exclude a neighborhood of every closed RHP zero of $\det D$, the convergence is uniform with respect to s. Since $P_i(s)=N_i(s)[D_i(s)]^{-1}$, $P(s)=N(s)[D(s)]^{-1}$, the same is true of P_i and P. □

One of the main difficulties with the graph topology is that the convergence of a sequence of matrices cannot be related in a simple way to the convergence of the component sequences. In other words, the graph topology on $\mathrm{mat}(R(s))$ is *not* the same as the product topology on $\mathrm{mat}(R(s))$ obtained from the graph topology on $R(s)$; in fact, neither contains the other. This was illustrated in Section I by means of two examples. The justification for the statements made in those examples is given next.

First consider the plant $P_\epsilon(s)$ of (1.8). This has the RCF

$$N_\epsilon=\begin{bmatrix}\lambda+\epsilon & \lambda-1\\ \lambda-2 & 1\end{bmatrix},\qquad D_\epsilon=D_0=\begin{bmatrix}\lambda-1 & \lambda\\ \lambda & \lambda-1\end{bmatrix},\qquad \lambda=\frac{1}{s+1}. \tag{2.13}$$

As $\epsilon\to 0$, N_ϵ approaches N_0, where

$$N_0=\begin{bmatrix}\lambda & \lambda-1\\ \lambda-2 & 1\end{bmatrix}. \tag{2.14}$$

Further, D_ϵ is independent of ϵ, so call it D_0. So, in the graph topology, $P_\epsilon=N_\epsilon D_\epsilon^{-1}$ converges to $P_0=N_0D_0^{-1}$, where the latter is given by (1.7). However, $(P_\epsilon)_{12}$ does *not* converge to $(P_0)_{12}$ [see (1.8)].

Now consider the plant P_ϵ of (1.9). This has the LCF

$$\tilde{d}_\epsilon=\frac{(s-1-\epsilon)(s-1+\epsilon)}{(s+1)^2},\qquad \tilde{N}_\epsilon=\begin{bmatrix}\dfrac{s-1-\epsilon}{s+1} & \dfrac{s-1+\epsilon}{s+1}\end{bmatrix} \tag{2.15}$$

whereas P_0 of (1.10) has the LCF

$$\tilde{d}_0=\frac{s-1}{s+1},\qquad \tilde{N}_0=[1\quad 1]. \tag{2.16}$$

From (1.9), we see that each component of P_ϵ converges in the graph topology to the corresponding component of P_0. Nevertheless, P_ϵ does not converge to P_0. Consider the set $\mathscr{R}=\{s\colon |s-1|\le 1/2\}$. Then P_0 has one pole inside $\mathscr{R}$, whereas P_ϵ has two poles inside $\mathscr{R}$ whenever $\epsilon\neq 0$. Hence from i) of Lemma 2.3, it follows that P_ϵ does not approach P_0.

Even though the graph topology is *not* a product topology in general, it is a product topology in two important special cases: for stable plants and for block-diagonal plants. The case of stable plants is covered by Lemma 2.2, which states that the graph topology, when restricted to the set $\mathrm{mat}(\mathscr{S})$, is the same as the topology induced by the norm (1.2), which is of course a product topology. For block-diagonal plants, we have the following result.

Lemma 2.4: Suppose $\{P_i\}$ is a sequence in $\mathrm{mat}(R(s))$, $P\in\mathrm{mat}(\mathscr{R}(\mathscr{S}))$, and suppose P_i, P are of the form

$$P_i=\begin{bmatrix}P_{1i} & & 0\\ & \ddots & \\ 0 & & P_{ni}\end{bmatrix}\quad\text{for all } i,\qquad P=\begin{bmatrix}P_1 & & 0\\ & \ddots & \\ 0 & & P_n\end{bmatrix} \tag{2.17}$$

where all partitions are of commensurate size. Then $P_i\to P$ in the graph topology if and only if $P_{li}\to P_l$ in the graph topology for each l in $\{1,\cdots,n\}$.

The proof of Lemma 2.4 is also given in Section IV.

To summarize, in this section we have defined a graph topology on the set $\mathrm{mat}(R(s))$ of (possibly) unstable plants, and have shown that convergence in the graph topology has a nice interpretation in terms of the ability to design stabilizing compensators and the continuity of the closed-loop response. In fact, the graph topology is the weakest topology on unstable plants such that feedback stability is robust. On the other hand, convergence of a sequence of matrices cannot be related in a simple way to the convergence of the component sequences, except in special cases. It is therefore desirable to have a simple test for convergence in the graph topology. This is provided by the graph metric introduced in Section III.

Finally, in some applications one may not wish to use the norms (1.1) and (1.2) to measure distances in $\mathscr{S}$ and $\mathrm{mat}(\mathscr{S})$. A careful examination of the proof of Theorem 2.1 reveals that any topology on $\mathrm{mat}(\mathscr{S})$ can be used [and not necessarily the one induced by the norm (1.2)], and the theorem remains valid, provided only that i) addition and multiplication in $\mathscr{S}$ are continuous, and ii) the set of units is open and inversion is continuous.

III. The Graph Metric

In Section II, we defined a topology on the set of unstable plants, and derived several qualitative properties (see Theorems 2.1 and 2.2). However, in order to obtain *quantitative* estimates of stability margins, it is desirable to have a quantitative measure of the disparity between two unstable plants. Towards this end, in this section we define a metric on the set $\mathrm{mat}(R(s))$ of possibly unstable plants, and show that the topology on $\mathrm{mat}(R(s))$ induced by this metric is the same as the graph topology of

Definition 2.1. For this reason, we refer to this metric as the *graph metric*. Robustness estimates for feedback stability based on the graph metric are given in Section IV.

A preliminary concept is needed to define the graph metric. Suppose $M(s)$ is a square rational matrix with the properties: i) $M(s)=M'(-s)$ for all s, where $'$ denotes the transpose; ii) $M(j\omega)$ is uniformly positive definite for all ω and is bounded as a function of ω; and iii) $M(s)$ is nonsingular for almost all s. Under these conditions, it is well known [12] that there exists a matrix A in $\text{mat}(\mathscr{S})$ such that $M(s)=A'(-s)A(s)$ and such that A is actually a *unit* of $\text{mat}(\mathscr{S})$(i.e., $A^{-1}\in \text{mat}(\mathscr{S})$). Such a matrix A is called a *spectral factor* of M and is unique to within left multiplication by an orthogonal matrix; that is, if A and B are both spectral factors of M, then there is an orthogonal matrix U such that $B=UA$.

Definition 3.1: A pair (N,D) is called a *normalized RCF* of a plant $P\in \text{mat}(R(s))$ if (N,D) is an RCF of P, and in addition

$$D'(-s)D(s)+N'(-s)N(s)=I,\qquad \text{for all } s. \tag{3.1}$$

Lemma 3.1: Every plant has a normalized RCF, which is unique to within right multiplication by an orthogonal matrix.

Proof: Suppose $P\in \text{mat}(R(s))$, and let (N_1,D_1) be any RCF of P. Define

$$M(s)=\begin{bmatrix} D_1'(-s) & N_1'(-s)\end{bmatrix}\begin{bmatrix} D_1(s)\\ N_1(s)\end{bmatrix}. \tag{3.2}$$

Then the coprimeness of N_1 and D_1 assures that M has a spectral factorization. Let A be a spectral factor of M and define $N=N_1A^{-1}$, $D=D_1A^{-1}$. Then (N,D) is a normalized RCF of P.

To show that (N,D) is unique except for the possibility of right multiplication by an orthogonal matrix, let (N_2,D_2) be any other RCF of P. Then $N_2=N_1V$, $D_2=D_1V$ for some unimodular matrix $V\in \text{mat}(\mathscr{S})$, and

$$M_2(s)=\begin{bmatrix} D_2'(-s) & N_2'(-s)\end{bmatrix}\begin{bmatrix} D_2(s)\\ N_2(s)\end{bmatrix}=V'(-s)M(s)V(s). \tag{3.3}$$

Thus, AV is a spectral factor of M_2; moreover, if B is any other spectral factor of M_2, then $B=UAV$ for some orthogonal matrix U. Hence, $(N_2B^{-1}, D_2B^{-1})=(N_1A^{-1}U^{-1}, D_1A^{-1}U^{-1})=(NU^{-1}, DU^{-1})$. □

Suppose (N,D) is a normalized RCF of P and let $A=\begin{bmatrix} D\\ N\end{bmatrix}$. If $R\in \text{mat}(\mathscr{S})$, then it is a ready consequence of (1.2) and (3.1) that $\|AR\|=\|R\|$. Thus, the map $R\to AR$ is an isometry on $\text{mat}(\mathscr{S})$. Similarly, if U is an orthogonal matrix, then $\|UR\|=\|R\|$ for all R in $\text{mat}(\mathscr{S})$.

We now define the graph metric.

Definition 3.2: Suppose $P_1,P_2\in \text{mat}(R(s))$ have the same dimensions, and let (N_i,D_i) be a normalized RCF of P_i for $i=1,2$. Define

$$A_i=\begin{bmatrix} D_i\\ N_i\end{bmatrix},\qquad i=1,2 \tag{3.4}$$

$$\delta(P_1,P_2)=\inf_{\|U\|\leqslant 1,\, U\in \text{mat}(\mathscr{S})}\|A_1-A_2U\| \tag{3.5}$$

$$d(P_1,P_2)=\max\{\delta(P_1,P_2),\delta(P_2,P_1)\}. \tag{3.6}$$

Then d is called the *graph metric* on $\text{mat}(R(s))$.

It is left to the reader to verify that $d(P_1,P_2)$ is a well-defined quantity, even though A_1,A_2 are only unique to within right multiplication by an orthogonal matrix. This is because multiplication by an orthogonal matrix does not change the norm.

Lemma 3.2: d is a metric on $\text{mat}(R(s))$ taking values in the interval $[0,1]$.

Proof: If $U=0$, then $\|A_1-A_2U\|=\|A_1\|=1$. Hence, $\delta(P_1,P_2)\leqslant 1$ and $d(P_1,P_2)\leqslant 1$ for all P_1,P_2. In proving that d is a metric, the only nonobvious part is the proof of the triangle inequality. Accordingly, let $P_1,P_2,P_3\in \text{mat}(R(s))$, and select $U,V\in \text{mat}(\mathscr{S})$ such that $\|U\|\leqslant 1$, $\|V\|\leqslant 1$, and

$$\|A_1-A_2U\|\leqslant \delta(P_1,P_2)+\epsilon \tag{3.7}$$

$$\|A_2-A_3V\|\leqslant \delta(P_2,P_3)+\epsilon \tag{3.8}$$

where ϵ is some positive number. Then $VU\in \text{mat}(\mathscr{S})$ and $\|VU\|\leqslant \|V\|\cdot\|U\|\leqslant 1$; moreover,

$$\begin{aligned}\|A_1-A_3VU\| &\leqslant \|A_1-A_2U\|+\|A_2U-A_3VU\|\\ &\leqslant \|A_1-A_2U\|+\|A_2-A_3V\|\cdot\|U\|\\ &\leqslant \delta(P_1,P_2)+\delta(P_2,P_3)+2\epsilon\end{aligned} \tag{3.9}$$

where we use the fact that $\|U\|\leqslant 1$. Now (3.9) shows that

$$\delta(P_1,P_3)\leqslant \delta(P_1,P_2)+\delta(P_2,P_3)+2\epsilon. \tag{3.10}$$

Since this is true for *every* $\epsilon>0$, it follows that

$$\delta(P_1,P_3)\leqslant \delta(P_1,P_2)+\delta(P_2,P_3). \tag{3.11}$$

By symmetry, we get the triangle inequality for $\delta(P_3,P_1)$, and hence for d. □

The main result of this section is Theorem 3.1 which states that the topology on $\text{mat}(R(s))$ induced by the graph metric is the same as the graph topology of Definition 2.1. To prove this theorem, we need the following technical fact.

Lemma 3.3: Suppose $P_1,P_2\in \text{mat}(R(s))$ have the same dimensions, and let A_1,A_2 be associated matrices in $\text{mat}(\mathscr{S})$ obtained from their normalized RCF's. Suppose $U,V\in \text{mat}(\mathscr{S})$ satisfy $\|U\|\leqslant 1$, $\|V\|\leqslant 1$, and $\|A_1-A_2U\|+\|A_2-A_1V\|<1$. Then U,V are unimodular.

Proof: We have

$$\begin{aligned}\|I-VU\|=\|A_1-A_1VU\| &\leqslant \|A_1-A_2U\|+\|A_2U-A_1VU\|\\ &\leqslant \|A_1-A_2U\|+\|A_2-A_1V\|<1.\end{aligned} \tag{3.12}$$

Hence, VU is unimodular, i.e., $\det(VU)$ is a unit of $\mathscr{S}$. Since $\det(VU)=\det V\cdot\det U$, it follows that both $\det V$ and $\det U$ are units of $\mathscr{S}$, i.e., that both U and V are unimodular in $\text{mat}(\mathscr{S})$. □

Theorem 3.1: A sequence $\{P_i\}$ in $\text{mat}(R(s))$ converges to P in the graph topology of Definition 2.1 if and only if $d(P_i,P)\to 0$.

Proof:

"If": Suppose $d(P_i,P)\to 0$. Then for each ϵ in the interval $(0,1/4)$ there is an integer i_0 such that $d(P_i,P)\leqslant\epsilon$ whenever $i\geqslant i_0$. For each i, there exist U_i,V_i in $\text{mat}(\mathscr{S})$ with norm at most one such that $\|A-A_iU_i\|\leqslant d(P_i,P)+\epsilon$, $\|A_i-AV_i\|\leqslant d(P_i,P)+\epsilon$. Now, if $i\geqslant i_0$, then

$$\|A-A_iU_i\|+\|A_i-AV_i\|\leqslant 4\epsilon<1. \tag{3.13}$$

Hence, by Lemma 3.3, $U_i,V_i\in\mathscr{U}$. Thus, if

$$A=\begin{bmatrix} D\\ N\end{bmatrix},\qquad A_i=\begin{bmatrix} D_i\\ N_i\end{bmatrix},$$

then (N_iU_i, D_iU_i) is an RCF of P_i, and $\|A-A_iU_i\|\leqslant 2\epsilon$ whenever $i\geqslant i_0$. It follows from Definition 2.1 that $P_i\to P$ in the graph topology.

"Only If": Suppose $P_i\to P$ in the graph topology, and let (N,D) be a normalized RCF of P. Then, by Lemma 2.1, there exist RCF's (N_i,D_i) of P_i such that

$$\begin{bmatrix} D_i\\ N_i\end{bmatrix}\to\begin{bmatrix} D\\ N\end{bmatrix}.$$

However, (N_i, D_i) need not be normalized. Let

$$M_i = \begin{bmatrix} D_i \\ N_i \end{bmatrix},$$

and suppose $M_i = A_i R_i$ where A_i corresponds to a normalized RCF and $R_i \in \mathscr{U}$. Now $\|M_i - A\| \to 0$, and $\|M_i\| = \|A_i R_i\| = \|R_i\|$ since A_i is an isometry. Hence, $\|R_i\| \to 1$. Define $U_i = R_i / \|R_i\|$. Then $\|U_i\| = 1$; moreover,

$$\begin{aligned}\|A - A_i U_i\| &\leq \|A - A_i R_i\| + \|A_i R_i - A_i U_i\| \\ &= \|A - M_i\| + \|R_i - U_i\| \qquad \text{since } A_i \text{ is an isometry} \\ &= \|A - M_i\| + |1 - \|R_i\|| \\ &\to 0 \qquad \text{as } i \to \infty. \end{aligned} \tag{3.14}$$

This shows that $\delta(P, P_i) \to 0$. To prove that $\delta(P_i, P) \to 0$, it is necessary to estimate $\|R_i^{-1}\|$. Let $\nu_i = \|A - M_i\|$. Then for all $x \in L_2$, we have

$$\|M_i x\| \geq \|Ax\| - \|(A - M_i)x\| \geq (1 - \nu_i)\|x\|.$$

Since

$$\|x\| = \|A_i x\| = \|M_i R_i^{-1} x\| \geq (1 - \nu_i)\|R_i^{-1} x\|,$$

it follows that $\|R_i^{-1}\| \leq 1/(1 - \nu_i)$ whenever $\nu_i < 1$. In particular, $\|R_i^{-1}\| \to 1$ as $i \to \infty$. Let $V_i = R_i^{-1}/\|R_i^{-1}\|$. Then $\|V_i\| = 1$, and

$$\begin{aligned}\|A_i - A V_i\| &= \|M_i R_i^{-1} - A V_i\| \\ &= \|M_i R_i^{-1} - A R_i^{-1} + A R_i^{-1} - A V_i\| \\ &\leq \|M_i - A\| \cdot \|R_i^{-1}\| + \|R_i^{-1} - V_i\| \\ &\to 0 \qquad \text{as } i \to \infty. \end{aligned} \tag{3.15}$$

Thus, $\delta(P_i, P) \to 0$ and the proof is complete. □

At present, the problem of computing the infimum in (3.5) is an unsolved problem in interpolation theory. Clearly,

$$\delta(P_i, P_2) \geq \inf_{U \in \operatorname{mat}(\mathscr{S})} \|A_1 - A_2 U\|. \tag{3.16}$$

The infimum in (3.16) can be computed using the results in [13], and provides a lower bound on $d(P_1, P_2)$ to go with the upper bound given next.

Lemma 3.4: Suppose (N, D) is a normalized RCF of P, that (N_1, D_1) is a (not necessarily normalized) RCF of P_1, and let

$$A = \begin{bmatrix} D \\ N \end{bmatrix}, \qquad M_1 = \begin{bmatrix} D_1 \\ N_1 \end{bmatrix}.$$

Suppose $\|A - M_1\| \triangleq \nu < 1$. Then

$$d(P, P_1) \leq 2\nu/(1 - \nu). \tag{3.17}$$

Proof: Suppose $M_1 = A_1 R_1$, where A_1 is a normalized RCF of P_1. Now $\|M_1\| \leq \|A\| + \|M_1 - A\| = 1 + \nu$. Hence $\|R_1\| = \|M_1\| \leq 1 + \nu$ and $\|R_1^{-1}\| \leq 1/(1 - \nu)$, as in the proof of Theorem 3.1. From (3.14) and (3.15) we get, successively,

$$\delta(P, P_1) \leq 2\nu \tag{3.18}$$

$$\delta(P_1, P) \leq 2\nu/(1 - \nu) \tag{3.19}$$

$$d(P, P_1) = \max\{\delta(P, P_1), \delta(P_1, P)\} \leq 2\nu/(1 - \nu). \tag{3.20}$$

This completes the proof. □

As mentioned earlier, $\operatorname{mat}(\mathscr{S})$ is an open subset of $\operatorname{mat}(R(s))$ in the graph topology (see the paragraph after the proof of Lemma 2.2). Thus, if $P \in \operatorname{mat}(\mathscr{S})$, then there is a number $c > 0$ such that $P_1 \in \operatorname{mat}(\mathscr{S})$ whenever $d(P, P_1) < c$. Estimating this number c leads to some robustness results, as discussed in the next section.

If P is an unstable plant, and particularly if P has $j\omega$-axis poles, then it is possible for *every* neighborhood of P to contain plants with a different number of RHP poles from P. For example, the reader can verify using Lemma 3.4 that

$$d\left(\frac{1}{s+\epsilon}, \frac{1}{s}\right) = 0(\epsilon), \qquad d\left(\frac{1}{s-\epsilon}, \frac{1}{s}\right) = 0(\epsilon). \tag{3.21}$$

Thus, every neighborhood of the unstable plant $1/s$ contains stable plants.

IV. Robustness Estimates for Feedback Stability

In this section, the graph metric introduced in Section III is used to derive some estimates for the robustness of feedback stability. We present at once the main result of this section, and defer the proof.

Theorem 4.1: Suppose the plant–compensator pair (P, C) is stable, and let $H(P, C)$ be the associated stable closed-loop transfer matrix defined in (1.3). Let $r = (1 + \|H(P, C)\|^2)^{1/2}$. Then the pair (P_1, C_1) is also stable whenever

$$\max\{d(P, P_1), d(C, C_1)\} \triangleq d < 1/(2 + 4r). \tag{4.1}$$

Moreover, if (4.1) holds, then

$$\|H(P_1, C_1) - H(P, C)\| \leq \frac{r\gamma(1 + r)}{1 - \gamma r} \tag{4.2}$$

where $\gamma = 4d/(1 - 2d)$.

Thus, Theorem 4.1 shows that if P_1, C_1 are sufficiently close to (P, C), then (P_1, C_1) is also stable, and $H(P_1, C_1)$ is close to $H(P, C)$. The measure of proximity of the perturbed pair (P_1, C_1) to the unperturbed pair (P, C) is provided by the larger of the graph metrics $d(P, P_1), d(C, C_1)$. The noteworthy features of Theorem 4.1 are the following: 1) $P_1(C_1)$ need not have the same number of RHP poles as $P(C)$; 2) $P_1(C_1)$ need not have the same dynamic order as $P(C)$; in fact, P_1 can be infinite-dimensional (representing a distributed parameter system), whereas P can be finite-dimensional (see Section VII for a discussion of distributed systems).

If P_1 has the same RHP poles as P, then it is in fact possible to derive *necessary and sufficient* conditions for robustness. This is done in Section V.

We now present a series of lemmas, culminating in the proof of Theorem 4.1.

Lemma 4.1: Suppose $P \in \operatorname{mat}(\mathscr{S})$, and let $r = (1 + \|P\|^2)^{1/2}$. Then $P_1 \in \operatorname{mat}(\mathscr{S})$ whenever $d(P, P_1) < 1/r$; moreover,

$$\|P_1 - P\| \leq [rd(1 + r)]/(1 - rd) \tag{4.3}$$

where d denotes $d(P, P_1)$.

Proof: Since P is stable, (P, I) is an RCF for P, although it may not be normalized. Let R be a spectral factor of $I + P'(-s)P(s)$. Then

$$R^{\dagger}(j\omega)R(j\omega) = I + P^{\dagger}(j\omega)P(j\omega) \tag{4.4}$$

which shows that $\|R\|^2 \leq 1 + \|P\|^2 = r^2$. Now (PR^{-1}, R^{-1}) is a normalized RCF of P. Further, since $d(P, P_1) < 1/r$, there exists a normalized RCF $(\bar{N}_1, \bar{D}_1)$ of P_1 and a matrix $U \in \operatorname{mat}(\mathscr{S})$ such that $\|U\| \leq 1$ and

$$\left\| \begin{bmatrix} \bar{D}_1 U \\ \bar{N}_1 U \end{bmatrix} - \begin{bmatrix} R^{-1} \\ PR^{-1} \end{bmatrix} \right\| \leq d(P, P_1) + \epsilon < 1/r. \tag{4.5}$$

In particular, $\|\overline{D}_1 U - R^{-1}\| < 1/r \leqslant 1/\|R\|$. This shows that $\overline{D}_1 U$ is a unimodular matrix, and by an easy calculation, that

$$\|(\overline{D}_1 U)^{-1} - R\| \leqslant \frac{\|R\|^2 \cdot \|\overline{D}_1 U - R^{-1}\|}{1 - \|R\| \cdot \|\overline{D}_1 U - R^{-1}\|} \leqslant \frac{r^2\gamma}{1 - r\gamma} \tag{4.6}$$

where $\gamma = d(P, P_1) + \epsilon$. Now, since $\overline{D}_1 U$ is unimodular, it follows that U is also unimodular, so that $P_1 = \overline{N}_1 U \cdot (\overline{D}_1 U)^{-1}$ is stable.

It only remains to estimate $\|P_1 - P\|$. For notational convenience, let $N = PR^{-1}$, $D = R^{-1}$, $N_1 = \overline{N}_1 U$, $D_1 = \overline{D}_1 U$. Then

$$\begin{aligned}\|P_1 - P\| &= \|N_1 D_1^{-1} - ND^{-1}\| \\ &\leqslant \|N_1 D_1^{-1} - ND_1^{-1}\| + \|ND_1^{-1} - ND^{-1}\| \\ &\leqslant \|N_1 - N\| \cdot \|D_1^{-1}\| + \|N\| \cdot \|D_1^{-1} - D^{-1}\|. \end{aligned} \tag{4.7}$$

From (4.6), we get

$$\|D_1^{-1}\| \leqslant \|D_1^{-1} - R\| + \|R\| \leqslant \frac{r^2\gamma}{1 - r\gamma} + r = \frac{r}{1 - r\gamma}. \tag{4.8}$$

Also, $\|N_1 - N\| \leqslant \gamma$ and $\|N\| \leqslant 1$, since $\left\|\begin{bmatrix} D \\ N \end{bmatrix}\right\| = 1$. Substituting all these into (4.7) gives

$$\|P_1 - P\| \leqslant \frac{r\gamma}{1 - r\gamma} + \frac{r^2\gamma}{1 - r\gamma} = \frac{r\gamma(1 + r)}{1 - r\gamma}. \tag{4.9}$$

Since $\gamma = d(P, P_1) + \epsilon$, (4.3) follows by letting ϵ approach 0. □

Lemma 4.2: Suppose S is a unimodular matrix. If P is a plant with an RCF (N, D), let SP denote the plant with RCF (N_s, D_s), where

$$\begin{bmatrix} D_s \\ N_s \end{bmatrix} = S \begin{bmatrix} D \\ N \end{bmatrix}. \tag{4.10}$$

Let P_1 be another plant with $d(P, P_1) < 1/2$. Then

$$d(SP, SP_1) \leqslant \frac{2\|S\| \cdot \|S^{-1}\| \cdot d(P, P_1)}{1 - \|S\| \cdot \|S^{-1}\| d(P, P_1)} \tag{4.11}$$

provided $\|S\| \cdot \|S^{-1}\| \cdot d(P, P_1) < 1$.

Proof: First we show that N_s, D_s are indeed right-coprime. Suppose X, Y satisfy

$$[Y \quad X] \begin{bmatrix} D \\ N \end{bmatrix} = I. \tag{4.12}$$

Then

$$[Y \quad X] \cdot S^{-1} \cdot \begin{bmatrix} D_s \\ N_s \end{bmatrix} = I. \tag{4.13}$$

Now suppose (N, D), (N_1, D_1) are normalized RCF's of P, P_1, and let

$$A = \begin{bmatrix} D \\ N \end{bmatrix}, \qquad A_1 = \begin{bmatrix} D_1 \\ N_1 \end{bmatrix}.$$

Select a U with $\|U\| \leqslant 1$ such that $\|A_1 U - A\| \leqslant d(P, P_1) + \epsilon < 1/2$. Then by Lemma 3.3, U is unimodular. Let R be a spectral factor of $A'(-s)S'(-s)S(s)A(s)$, and define

$$\begin{bmatrix} \overline{D}_s \\ \overline{N}_s \end{bmatrix} = S \begin{bmatrix} D \\ N \end{bmatrix} R^{-1} = SAR^{-1}. \tag{4.14}$$

Then $(\overline{N}_s, \overline{D}_s)$ is a normalized RCF of the plant SP. Now $SA_1 UR^{-1}$ is of the form $\begin{bmatrix} \overline{D}_1 \\ \overline{N}_1 \end{bmatrix}$, where $(\overline{N}_1, \overline{D}_1)$ is an RCF of the plant SP_1 (observe that both U and R^{-1} are unimodular). Moreover,

$$\begin{aligned}\left\| SA_1 UR^{-1} - \begin{bmatrix} \overline{D}_s \\ \overline{N}_s \end{bmatrix} \right\| &= \|SA_1 UR^{-1} - SAR^{-1}\| \\ &\leqslant \|S\| \cdot \|A_1 U - A\| \cdot \|R^{-1}\| \\ &\leqslant \|S\| \cdot \|R^{-1}\| \cdot [d(P, P_1) + \epsilon]. \end{aligned} \tag{4.15}$$

Now note that

$$\|R^{-1}\| = \|AR^{-1}\| = \|S^{-1} \cdot SAR^{-1}\| = \|S^{-1}\| \tag{4.16}$$

where we use the fact that both A and SAR^{-1} are isometries. Thus, (4.15) and (4.16) show that

$$\|SA_1 UR^{-1} - SAR^{-1}\| \leqslant \|S\| \cdot \|S^{-1}\| \cdot [d(P, P_1) + \epsilon]. \tag{4.17}$$

Now, SAR^{-1} gives a normalized RCF of the plant SP, while $SA_1 UR^{-1}$ gives a (not necessarily normalized) RCF of the plant SP_1. If we apply Lemma 3.4 together with (4.17) and let ϵ approach zero, we get (4.11). □

Proof of Theorem 4.1: Consider the feedback system of Fig. 1, and let

$$G = \begin{bmatrix} C & 0 \\ 0 & P \end{bmatrix}, \qquad G_1 = \begin{bmatrix} C_1 & 0 \\ 0 & P_1 \end{bmatrix} \tag{4.18}$$

where P_1, C_1 are perturbed versions of P, C. Clearly, if (N_p, D_p), (N_c, D_c) are normalized RCF's of P and C, respectively, then

$$\left(\begin{bmatrix} N_c & 0 \\ 0 & N_p \end{bmatrix}, \begin{bmatrix} D_c & 0 \\ 0 & D_p \end{bmatrix} \right) \tag{4.19}$$

is a normalized RCF of G. Thus, $d(G, G_1) \leqslant \max\{d(P, P_1), d(C, C_1)\}$.

Now note that

$$H(P, C) = (I + FG)^{-1} \tag{4.20}$$

where

$$F = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}. \tag{4.21}$$

Thus, if (N_g, D_g) is an RCF of G, then

$$\begin{bmatrix} D_h \\ N_h \end{bmatrix} = \begin{bmatrix} I & F \\ I & 0 \end{bmatrix} \begin{bmatrix} D_g \\ N_g \end{bmatrix} \tag{4.22}$$

is an RCF of H (in other words, $H = (I + FG)^{-1} = D_g(D_g + FN_g)^{-1}$). Hence, the plant H is of the form SG, where S is the unimodular matrix

$$S = \begin{bmatrix} I & F \\ I & 0 \end{bmatrix}. \tag{4.23}$$

An easy calculation shows that $\|S\| = \|S^{-1}\| = \sqrt{2}$. Hence, from Lemma 4.2,

$$\begin{aligned} d(H(P_1, C_1), H(P, C)) &= d\left[(I + FG_1)^{-1}, (I + FG)^{-1}\right] \\ &\leqslant 4d(G, G_1)/[1 - 2d(G, G_1)]. \end{aligned} \tag{4.24}$$

Now suppose the pair (P,C) is stable, and let $r=(1+\|H(P,C)\|^2)^{1/2}$. By Lemma 4.1, if the quantity on the right side of (4.24) is less than $1/r$, then $H(P_1,C_1)$ is also stable. Since $d(G,G_1)\leqslant\max\{d(P,P_1),d(C,C_1)\}\triangleq d$, the right side of (4.24) is no larger than

$$\gamma\triangleq 4d/(1-2d). \tag{4.25}$$

Thus, from Lemma 4.1, a sufficient condition for (P_1,C_1) to be stable is $\gamma r<1$, or equivalently

$$d<1/(2+4r). \tag{4.26}$$

This is the same as (4.1). If (4.26) holds, then (4.3) with γ replacing d leads to (4.2). □

We now examine the case where a nominal pair (P,C) is not necessarily stable.

Lemma 4.3: The map $(P,C)\to H(P,C)$ is continuous in the graph metric. Specifically, whenever $d(P,P_1)<1/2, d(C,C_1)<1/2$, we have

$$d(H(P,C),H(P_1,C_1))\leqslant 4d/[1-2d] \tag{4.27}$$

where $d=\max\{d(P,P_1),d(C,C_1)\}$.

Proof: See (4.24). □

We are now in a position to give a proof of Theorem 2.2. For convenience, the theorem is restated here.

Theorem 2.2: Suppose $\{P_i\},\{C_i\}$ are sequences in $\mathrm{mat}(R(s))$, that $P,C\in\mathrm{mat}(R(s))$, and that (P,C) is stable. Suppose $P_i\to P$, $C_i\to C$ in the graph topology; then (P_i,C_i) is stable for large enough i and $H(P_i,C_i)\to H(P,C)$ in $\mathrm{mat}(\mathscr{S})$. Conversely, suppose (P_i,C_i) is stable and $H(P_i,C_i)\to H(P,C)$; then $P_i\to P$, $C_i\to C$ in the graph topology.

Proof: The second sentence is already proved in Theorem 4.1. To prove the last sentence, define

$$G=\begin{bmatrix} C & 0\\ 0 & P\end{bmatrix},\qquad G_i=\begin{bmatrix} C_i & 0\\ 0 & P_i\end{bmatrix} \tag{4.28}$$

and let F be as in (4.21). The hypothesis is that $H(P_i,C_i)\to H(P,C)$, or that $(I+FG_i)^{-1}\to(I+FG)^{-1}$ in $\mathrm{mat}(\mathscr{S})$. Now

$$H(G_i,F)=\begin{bmatrix}(I+G_iF)^{-1} & -G_i(I+FG_i)^{-1}\\ F(I+G_iF)^{-1} & (I+FG_i)^{-1}\end{bmatrix} \tag{4.29}$$

from (1.3). Since $(I+FG_i)^{-1}\to(I+FG)^{-1}$, it follows that

$$\begin{aligned}(I+FG_i)^{-1}F=F(I+G_iF)^{-1}&\to(I+FG)^{-1}F\\ &=F(I+GF)^{-1}.\end{aligned}$$

Next, since

$$G_i(I+FG_i)^{-1}=F^{-1}\left[I-(I+FG_i)^{-1}\right] \tag{4.30}$$

it follows that $G_i(I+FG_i)^{-1}\to G(I+FG)^{-1}$. Finally, since

$$(I+G_iF)^{-1}=I-G_i(I+FG_i)^{-1}F \tag{4.31}$$

it follows that $(I+G_iF)^{-1}\to(I+GF)^{-1}$. In other words, $H(G_i,F)\in\mathrm{mat}(\mathscr{S})$, and $\|H(G_i,F)-H(G,F)\|\to 0$. If we now apply Theorem 2.1 with P,P_i,C replaced by G,G_i,F, respectively, we can conclude that $G_i\to G$ in the graph topology.

The trick now is to prove from this that $P_i\to P$, $C_i\to C$ in the graph topology. As the example in (1.7) and (1.8) shows, this is not automatic. Let (N_{1i},D_{1i}), (N_{2i},D_{2i}) be normalized RCF's of C_i,P_i, and let (N_1,D_1), (N_2,D_2) be normalized RCF's of C,P, respectively. Then normalized RCF's for G_i and G can be formed as in (4.19). Since $G_i\to G$ in the graph topology, there exists a sequence $\{U_i\}$ of unimodular matrices such that

$$\begin{bmatrix} N_{1i} & 0\\ 0 & N_{2i}\\ D_{1i} & 0\\ 0 & D_{2i}\end{bmatrix}\begin{bmatrix} U_{1i} & U_{2i}\\ U_{3i} & U_{4i}\end{bmatrix}\to\begin{bmatrix} N_1 & 0\\ 0 & N_2\\ D_1 & 0\\ 0 & D_2\end{bmatrix}\quad\text{in }\mathrm{mat}(\mathscr{S}) \tag{4.32}$$

where U_i is partitioned in the obvious way. Since the topology on $\mathrm{mat}(\mathscr{S})$ is the product topology, each block in the partitioned matrix on the left side of (4.32) converges to the corresponding block on the right side of (4.32). Thus,

$$\begin{bmatrix} N_{1i}U_{2i}\\ D_{1i}U_{2i}\end{bmatrix}\to\begin{bmatrix}0\\0\end{bmatrix},\qquad\begin{bmatrix} N_{2i}U_{3i}\\ D_{2i}U_{3i}\end{bmatrix}\to\begin{bmatrix}0\\0\end{bmatrix}. \tag{4.33}$$

Since $\begin{bmatrix} N_{1i}\\ D_{1i}\end{bmatrix},\begin{bmatrix} N_{2i}\\ D_{2i}\end{bmatrix}$ are all isometries, it follows that $U_{2i}\to 0$, $U_{3i}\to 0$. Now let A_i (resp. A) denote the rectangular matrix on the left (resp. right) side of (4.32), and define

$$B_i=A-A_iU_i \tag{4.34a}$$

$$W_i=\begin{bmatrix} 0 & U_{2i}\\ U_{3i} & 0\end{bmatrix} \tag{4.34b}$$

$$c_i=\|U_i-W_i\|=\left\|\begin{bmatrix} U_{1i} & 0\\ 0 & U_{2i}\end{bmatrix}\right\| \tag{4.34c}$$

$$\overline{U}_i=\frac{1}{c_i}\begin{bmatrix} U_{1i} & 0\\ 0 & U_{2i}\end{bmatrix}. \tag{4.34d}$$

Then $c_i\to 1$ since $\|W_i\|\to 0$ and $\|U_i\|\to 1$. Therefore,

$$\begin{aligned}\|A-A_i\overline{U}_i\|&=\|A-A_i(U_i-W_i)/c_i\|\\ &\leqslant\|A-A_iU_i/c_i\|+\|A_iW_i/c_i\|\\ &\leqslant\|A-A_iU_i\|+\left\|A_iU_i\left(1-\frac{1}{c_i}\right)\right\|+\|A_iW_i/c_i\|\\ &\to 0\end{aligned}$$

since A_i is an isometry, $\|U_i\|$ is bounded, $c_i\to 1$, and $\|W_i\|\to 0$. Observe also that $\|\overline{U}_i\|=1$. Since $G_i\to G$ in the graph metric, there exists a sequence V_i of matrices such that $\|V_i\|\leqslant 1$ and $\|AV_i-A_i\|\to 0$. Since $\|A-A_i\overline{U}_i\|+\|AV_i-A_i\|\to 0$, it is less than 1 for large enough i. By Lemma 3.3, it now follows that $\overline{U}_i,V_i$ are unimodular for large enough i. This implies that U_{1i},U_{4i} are unimodular for large enough i, and that

$$\begin{bmatrix} N_{1i} & U_{1i}\\ D_{1i} & U_{1i}\end{bmatrix}\to\begin{bmatrix} N_1\\ D_1\end{bmatrix},\qquad\begin{bmatrix} N_{2i} & U_{4i}\\ D_{2i} & U_{4i}\end{bmatrix}\to\begin{bmatrix} N_2\\ D_2\end{bmatrix} \tag{4.35}$$

which shows that $P_i\to P$, $C_i\to C$ in the graph topology. □

Proof of Lemma 2.4: In the preceding proof of Theorem 2.2 it was shown that

$$\begin{bmatrix} C_i & 0\\ 0 & P_i\end{bmatrix}\to\begin{bmatrix} C & 0\\ 0 & P\end{bmatrix}\Rightarrow C_i\to C,\quad P_i\to P. \tag{4.36}$$

The converse of the above implication is easy to establish: Suppose $C_i\to C$, $P_i\to P$. Let (N_c,D_c), (N_p,D_p) be RCF's of C and P, respectively, and select sequences of RCF's (N_{ci},D_{ci}) of C_i, (N_{pi},D_{pi}) of P_i such that

$$\begin{bmatrix} D_{ci}\\ N_{ci}\end{bmatrix}\to\begin{bmatrix} D_c\\ N_c\end{bmatrix},\qquad\begin{bmatrix} D_{pi}\\ N_{pi}\end{bmatrix}\to\begin{bmatrix} D_p\\ N_p\end{bmatrix}. \tag{4.37}$$

Then

$$\begin{bmatrix} D_{ci} & 0 \\ 0 & D_{pi} \end{bmatrix} \to \begin{bmatrix} D_c & 0 \\ 0 & D_p \end{bmatrix}, \quad \begin{bmatrix} N_{ci} & 0 \\ 0 & N_{pi} \end{bmatrix} \to \begin{bmatrix} N_c & 0 \\ 0 & N_p \end{bmatrix} \tag{4.38}$$

which shows that

$$\begin{bmatrix} C_i & 0 \\ 0 & P_i \end{bmatrix} \to \begin{bmatrix} C & 0 \\ 0 & P \end{bmatrix}. \tag{4.39}$$

This completes the proof of Lemma 2.4 in the case of block-diagonal matrices with two blocks. The case of more than two blocks now follows readily by induction. □

The robustness estimates of Theorem 4.1 represent a good beginning, but are quite conservative. For instance, in Lemma 4.2, if we substitute $S = I$, the identity operator, then the right side of (4.11) becomes

$$\frac{2d(P,P_i)}{1-d(P,P_1)} > 2d(P,P_1). \tag{4.40}$$

Compared to the true value of $d(SP, SP_1)$ (which is $d(P,P_1)$ since $S = I$), this bound is not very tight. This looseness is carried over into (4.1). Another failure of Theorem 4.1 is in not accommodating some sort of frequency-dependent weighting in computing the various distances. These problems are left for future research.

V. Specialized Robustness Results

In this section, we study the robustness of feedback systems when the perturbed plant has the same RHP poles as the unperturbed plant. Both additive as well as multiplicative perturbations are studied. By restricting to this special class of perturbations, we are able to derive *necessary and sufficient conditions* for robustness. The current results are weaker than existing ones in the case of multiplicative perturbations, since other authors [23] give necessary and sufficient conditions for robust stability when the perturbed plant has the same *number* of RHP poles as the unperturbed plant, although possibly at different locations. However, the results in the case of additive perturbations are new.

The main tool used in this section is the following.

Lemma 5.1: Suppose $F \in \mathrm{mat}(\mathscr{S})$. Then $I + RF$ is unimodular for all $R \in \mathrm{mat}(\mathscr{S})$ with $\|R\| \leqslant r$ if and only if $\|F\| < 1/r$.

Proof:

"If": Suppose $\|F\| < 1/r$. Then, whenever $\|R\| \leqslant r$, we have $\|FR\| < 1$, which implies that $I + FR$ is unimodular.

"Only If": This part is constructive. Suppose $\|F\| \geqslant 1/r$; we construct an R with $\|R\| \leqslant r$ such that $I + RF$ is not unimodular. Let $\|A\|_2$ denote the matrix norm defined by

$$\|A\|_2 = \left[\lambda_{\max}(A^{\dagger}A)\right]^{1/2}. \tag{5.1}$$

Then

$$\|F\| = \sup_{\omega} \|F(j\omega)\|_2. \tag{5.2}$$

Hence, if $\|F\| \geqslant r^{-1}$, then either $\|F(\infty)\|_2 \geqslant r^{-1}$ or else $\|F(j\omega)\|_2 \geqslant r^{-1}$ for some ω. Suppose first that $\|F(\infty)\|_2 \geqslant r^{-1}$. Then, since $F(\infty)$ is a real matrix, there exist *real* vectors u, v such that $\|v\|_2 = 1$, $\|u\|_2 \geqslant r^{-1}$, and $Fv = u$. Let $c = \|u\|_2$ and let R equal the constant matrix $-vu'/c^2$. Then $\|R\| = c^{-1} \leqslant r$; moreover $I - RF(\infty)$ is singular, since $RF(\infty)v = -v$. Thus, $I + RF$ is not unimodular. Next, suppose $\|F(\infty)\|_2 < r^{-1}$, but $\|F(j\omega)\|_2 \geqslant r^{-1}$ for some finite ω. The only additional complication is that $F(j\omega)$ might be a complex matrix. If $\omega = 0$, then $F(j0)$ is real and the above proof applies. If $\omega > 0$, select (possibly complex) vectors v, u such that $\|v\|_2 = 1$, $\|u\|_2 \geqslant r^{-1}$, and $Fv = u$. Suppose

$$v = \begin{bmatrix} v_1 e^{j\theta_1} \\ \vdots \\ v_m e^{j\theta_m} \end{bmatrix}, \quad u = \begin{bmatrix} u_1 e^{j\phi_1} \\ \vdots \\ u_n e^{j\phi_n} \end{bmatrix} \tag{5.3}$$

where the v_i, u_i are real, and the θ_i, ϕ_i lie in $[-\pi, 0)$. Choose

$$R(s) = -\begin{bmatrix} \frac{s-\alpha_1}{s+\alpha_1} & & 0 \\ & \ddots & \\ 0 & & \frac{s-\alpha_m}{s+\alpha_m} \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix} [u_1 \cdots u_n] \cdot \begin{bmatrix} \frac{s-\beta_1}{s+\beta_1} & & 0 \\ & \ddots & \\ 0 & & \frac{s-\beta_n}{s+\beta_n} \end{bmatrix} \frac{1}{\|u\|_2^2} \tag{5.4}$$

where the constants α_i, β_i are adjusted so that

$$\arg\left(\frac{j\omega - \alpha_i}{j\omega + \alpha_i}\right) = \theta_i, \quad \arg\left(\frac{j\omega - \beta_i}{j\omega + \beta_i}\right) = \phi_i, \qquad \text{for all } i. \tag{5.5}$$

Then $\|R\| \leqslant r$ and $(I + RF)(j\omega)$ is singular, so that $I + RF$ is not unimodular. □

In what follows, we consider the robustness of the feedback system of Fig. 1 when P is perturbed to either $(I + L)P$ where $L \in \mathrm{mat}(\mathscr{S})$ and $\|L\| < 1$ or to $P + L$, $L \in \mathrm{mat}(\mathscr{S})$. The former is referred to as a multiplicative perturbation and the latter as an additive perturbation. In both cases, both Q and P have the same RHP poles.

Theorem 5.1: Suppose the pair (P, C) is stable. Then C stabilizes every Q of the form $(I + L)P$ for all $L \in \mathrm{mat}(\mathscr{S})$ with $\|L\| \leqslant r < 1$, if and only if $\|PC(I + PC)^{-1}\| < r^{-1}$.

Proof: Let (N_p, D_p) be any RCF of P. Then $((I + L)N_p, D_p)$ is an RCF of Q. Since (P, C) is stable, it follows from [10] that there exists an LCF $(\tilde{D}_c, \tilde{N}_c)$ of C such that $\tilde{D}_c D_p + \tilde{N}_c N_p = I$. If P is replaced by Q, this "return difference" matrix becomes $\tilde{D}_c D_p + \tilde{N}_c(I + L)N_p = I + \tilde{N}_c L N_p$. Thus, C stabilizes every Q of the form $(I + L)P$ if and only if $I + \tilde{N}_c L N_p$ is unimodular for all L in $\mathrm{mat}(\mathscr{S})$ with $\|L\| \leqslant r$. Now $\det(I + \tilde{N}_c L N_p) = \det(I + L N_p \tilde{N}_c)$; hence $I + \tilde{N}_c L N_p$ is unimodular if and only if $I + L N_p \tilde{N}_c$ is. Now from Lemma 5.1, it follows that $I + L N_p \tilde{N}_c$ is unimodular whenever $\|L\| \leqslant r$ if and only if $\|N_p \tilde{N}_c\| < r^{-1}$. It is easily verified that $N_p \tilde{N}_c = PC(I + PC)^{-1}$. □

Theorem 5.2: Suppose (P, C) is stable. Then C stabilizes every Q of the form $P + L$ for all $L \in \mathrm{mat}(\mathscr{S})$ with $\|L\| \leqslant r$, if and only if $\|C(I + PC)^{-1}\| < r^{-1}$.

Sketch of Proof: $P + L$ has the RCF $(N_p + LD_p, D_p)$. So the return difference matrix is $\tilde{D}_c D_p + \tilde{N}_c(N_p + LD_p) = I + \tilde{N}_c L D_p$. Now $\det(I + \tilde{N}_c L D_p) = \det(I + L D_p \tilde{N}_c)$. So we require $\|D_p \tilde{N}_c\| < r^{-1}$. Finally, $D_p \tilde{N}_c = C(I + PC)^{-1}$. □

VI. Application to Singularly Perturbed Systems

In this section, we apply the theory developed in Section II to analyze the continuity of singularly perturbed systems in the graph topology. Consider the system

$$\begin{bmatrix} \dot{x} \\ \epsilon \dot{z} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \tag{6.1}$$

$$(P_\epsilon)$$

$$y = [C_1 \quad C_2] \begin{bmatrix} x \\ z \end{bmatrix} + D_1 u \tag{6.2}$$

where $\epsilon > 0$ and A_{22} is a Hurwitz matrix (i.e., all eigenvalues of A_{22} have negative real parts). Let P_0 denote the system obtained from (6.1), (6.2) by substituting $\epsilon = 0$, namely

$$\dot{x} = Ax + Bu, \quad y = Cx + Du \qquad (P_0) \tag{6.3}$$

where

$$\begin{aligned} A &= A_{11} - A_{12}A_{22}^{-1}A_{21}, & B &= B_1 - A_{12}A_{22}^{-1}B_2 \\ C &= C_1 - C_2A_{22}^{-1}A_{21}, & D &= D_1 - C_2A_{22}^{-1}B_2. \end{aligned} \tag{6.4}$$

The question studied here is: does P_ϵ approach P_0 in the graph topology as $\epsilon \to 0$?[7] The motivation for this study is the well-known fact (see, e.g., [16]) that, if P_0 is unstable and a controller C stabilizes P_0, then C may not in general stabilize P_ϵ for small enough ϵ. In this context, Theorem 2.1 is significant. Applied to the problem at hand, it implies that if $P_\epsilon \to P_0$ in the graph topology, then *every* controller C that stabilizes P_0 also stabilizes P_ϵ for small enough ϵ, and the resulting closed-loop transfer matrix $H(P_\epsilon, C)$ approaches $H(P_0, C)$. On the other hand, if P_ϵ does not approach P_0 and C stabilizes P_0, then one of two things happens: either C does not stabilize P_ϵ for small enough ϵ, or else C does stabilize P_ϵ for small enough ϵ, but $H(P_\epsilon, C)$ does not approach $H(P_0, C)$.

One could also explore the possibility of using an ϵ-dependent family of controllers C_ϵ such that C_ϵ stabilizes P_ϵ for small enough ϵ, and $H(P_\epsilon, C_\epsilon) \to H(P_0, C_0)$ as $\epsilon \to 0$. This situation is addressed by Theorem 2.2. In the present situation, this theorem implies that if such a family $\{C_\epsilon\}$ exists, then $P_\epsilon \to P_0$, $C_\epsilon \to C_0$ as $\epsilon \to 0$. Thus, if P_ϵ does not converge to P_0, then one cannot even find an ϵ-dependent family C_ϵ such that (P_0, C_0) is stable and $H(P_\epsilon, C_\epsilon) \to H(P_0, C_0)$. On the other hand, if $P_\epsilon \to P_0$, then one can get by with an "ϵ-independent" controller.[8]

We state at once the main result of this section and devote the rest of the section to its proof.[9]

Theorem 6.1: Suppose the system (6.3) is stabilizable and detectable, and that A_{22} is Hurwitz. Then P_ϵ approaches P_0 in the graph topology as $\epsilon \to 0$ if $C_2(sI - A_{22})^{-1}B_2 \equiv 0$.

Many previously known results can be obtained as ready consequences of Theorem 6.1. For instance, it is shown in [17] that if the system (6.3) is stabilized by state feedback, then the same state feedback stabilizes the system (6.1), (6.2) for small enough ϵ. In the present setup, this corresponds to the case where $y = x$ and the stabilizing controller for P_0 is just a static gain (call it K). Since $y = x$, we have $C_1 = I$, $C_2 = 0$, hence $C_2(sI - A_{22})^{-1}B_2 \equiv 0$. Thus, Theorem 6.1 implies that $P_\epsilon \to P_0$, and Theorem 2.1 now implies that K also stabilizes P_ϵ for small enough ϵ. We also get an added bit of information from Theorem 2.1 that is not discussed in [17], namely that $H(P_\epsilon, K)$ converges to $H(P_0, K)$. Similarly, it is shown in [16] that if $B_2 = 0$ or $C_2 = 0$, and if P_0 is stabilized using an observer-controller scheme, the same scheme stabilizes P_ϵ for small enough ϵ. This too can be deduced from Theorems 6.1 and 2.1. More generally, Theorem 6.1 shows that if $C_2(sI - A_{22})^{-1}B_2 \equiv 0$, then *every* controller that stabilizes P_0 also stabilizes P_ϵ for small enough ϵ. No details need be known about the configuration of the controller. Thus, Theorem 6.1 (together with Theorem 2.1) gives means of unifying several known results.

Now we move toward a proof of Theorem 6.1. Lemma 6.1 below is of independent interest, as it provides a method for obtaining a left-coprime factorization of a system described in state-space form.[10] Since finding an RCF of a plant $P(s)$ is equivalent to finding an LCF for $P'(s)$, the same lemma can also be used for finding RCF's.

[7] I thank Prof. M. Suzuki of Nagoya University for drawing my attention to the possibility of applying this topological approach to singularly perturbed systems.

[8] I thank Prof. A. Willsky of M.I.T. for raising this issue.

[9] I thank Prof. H. Khalil and P. Kokotovic for pointing out an error in an earlier version of this theorem.

[10] The expression for the left-coprime factorization is due to my colleague D. Aplevich, while the main idea of the proof is due to my student M. McIntyre.

Lemma 6.1: Consider a system of the form (6.3), and suppose it is stabilizable and detectable. Select a matrix F such that $A - FC \triangleq \bar{A}$ is Hurwitz, and define

$$\tilde{N}(s) = C(sI - \bar{A})^{-1}(B - FD) + D \tag{6.5a}$$

$$\tilde{D}(s) = I - C(sI - \bar{A})^{-1}F. \tag{6.5b}$$

Then $(\tilde{D}, \tilde{N})$ is a left-coprime factorization of $P_0(s) = C(sI - A)^{-1}B + D$.

Proof: First we show that $P_0(s) = [\tilde{D}(s)]^{-1}\tilde{N}(s)$ by establishing that $\tilde{D}(s)y(s) = \tilde{N}(s)u(s)$. From (6.3), we get

$$\dot{x} = Ax + Bu = \bar{A}x + Fy + (B - FD)u \tag{6.6}$$

$$x(s) = (sI - \bar{A})^{-1}[Fy(s) + (B - FD)u(s)] \tag{6.7}$$

$$y(s) = C(sI - \bar{A})^{-1}[Fy(s) + (B - FD)u(s)] + Du(s). \tag{6.8}$$

It only remains to show that $\tilde{D}, \tilde{N}$ are left-coprime. This is equivalent to showing that the matrix $[\tilde{D}(s) \quad \tilde{N}(s)]$ has full row rank whenever $\operatorname{Re} s \geqslant 0$ and at $s = \infty$. At $s = \infty$, we have $[\tilde{D}(\infty) \quad \tilde{N}(\infty)] = [I \quad D]$, so let us examine the case where $s \in C_+$. If $|\tilde{D}(s)| \neq 0$ for some s, then $[\tilde{D}(s) \quad \tilde{N}(s)]$ has full rank; so it is only necessary to examine those $s \in C_+$ at which $|\tilde{D}(s)| = 0$ to ensure that $[\tilde{D}(s) \quad \tilde{N}(s)]$ has full rank at these values of s as well.

In order to do this, it is first established that any C_+-zero of $|\tilde{D}(\cdot)|$ must be an eigenvalue of A. This is done by showing that

$$\tilde{D}(s) = \left[I + C(sI - A)^{-1}F\right]^{-1}. \tag{6.9}$$

To prove (6.9) let E denote $sI - A$. Then from (6.5b)

$$\tilde{D} = I - C(E + FC)^{-1}F. \tag{6.10}$$

We want to show that

$$\tilde{D} = [I + CE^{-1}F]^{-1}. \tag{6.11}$$

Thus, the claim is that

$$\left[I - C(E + FC)^{-1}F\right](I + CE^{-1}F) = I$$

or, equivalently

$$I + CE^{-1}F - C(E + FC)^{-1}F - C(E + FC)^{-1}FCE^{-1}F = I \tag{6.12}$$

$$C\left[E^{-1} - (E + FC)^{-1} - (E + FC)^{-1}FCE^{-1}\right]F = 0. \tag{6.13}$$

The quantity inside the brackets equals

$$\begin{aligned} &\left[I - (E + FC)^{-1}E - (E + FC)^{-1}FC\right]E^{-1} \\ &\qquad = \left[I - (E + FC)^{-1}(E + FC)\right]E^{-1} = 0. \end{aligned} \tag{6.14}$$

Thus, (6.9) holds. Now let $m(s) = |I + C(sI - A)^{-1}F|$. Then (6.9) shows that every (not necessarily C_+) zero of $|\tilde{D}(\cdot)|$ must be a pole of $m(\cdot)$. But $m(\cdot)$ is analytic except at the eigenvalues of A. Thus, every (not necessarily C_+) zero of $|\tilde{D}(\cdot)|$ must be an eigenvalue of A. Further, if $|\tilde{D}(s_0)| = 0$, then the multiplicity of s_0 as a zero of $|\tilde{D}(\cdot)|$ is no larger than its multiplicity as an eigenvalue of A.

To conclude the proof, suppose $s_0 \epsilon C_-$ and that $|\tilde{D}(s_0)| = 0$. By the preceding discussion, s_0 is an eigenvalue of A, say of multiplicity μ. Then the multiplicity of s_0 as a zero of $|\tilde{D}(\cdot)|$ is no larger than μ, and by the stabilizability and detectability assumptions, the McMillan degree of s_0 as a pole of $P(\cdot)$ equals μ; that is, there exists a minor $p\binom{J}{K}$ of P such that $p\binom{J}{K}(\cdot)$ has a pole of order μ at s_0. By [24, p. 50], there is a one-to-one correspondence between the minors of P and the minors of $[\tilde{D} \quad \tilde{N}]$. In fact,

$$p\binom{J}{K}=\left|\tilde{D}\left[\begin{matrix}\cdot\\ M\backslash J\end{matrix}\right] \;\vdots\; \tilde{N}\left[\begin{matrix}\cdot\\ K\end{matrix}\right]\right|\cdot|\tilde{D}|^{-1} \tag{6.15}$$

where $M=\{1,\cdots,m\}$ is the set of all columns of D. Let $F=[\tilde{D} \quad \tilde{N}]$ and let $f(J,K)$ denote the minor in (6.15), so that $p\binom{J}{K}=f(J,K)\ |\tilde{D}|$. Now $p\binom{J}{K}$ has a pole at s_0 of order μ, while $\tilde{D}^{-1}$ has a pole at s_0 or order no larger than μ. Hence, $f(J,K)(s_0)\neq 0$, i.e., $F(s_0)$ has full row rank. Since this argument can be repeated at *all* C_+-zeros of $|D(\cdot)|$, it follows that $\tilde{D}, \tilde{N}$ are left-coprime. □

Proof of Theorem 6.1: Suppose $C_2(sI-A_{22})^{-1}B_2\equiv 0$ select F such that $A-FC$ is Hurwitz, and define $\tilde{N}, \tilde{D}$ by (6.4) and (6.5). Then $(\tilde{D},\tilde{N})$ is an LCF of P_0, by Lemma 6.1. To obtain an LCF of P_ϵ, define $F_0=\begin{bmatrix}F\\0\end{bmatrix}$, and consider

$$A_\epsilon=\begin{bmatrix}A_{11}-FC_1 & A_{12}-FC_2\\ A_{21}/\epsilon & A_{22}/\epsilon\end{bmatrix}. \tag{6.16}$$

From [17], the eigenvalues of $\bar{A}_\epsilon$ are asymptotically equal to those of A_{22}/ϵ, plus those of

$$\begin{aligned}A_{11}-FC_1-&(A_{12}-FC_2)A_{22}^{-1}A_{21}\\ &=(A_{11}-A_{12}A_{22}^{-1}A_{21})-F(C_1-C_2A_{22}^{-1}A_{21})\\ &=A-FC.\end{aligned} \tag{6.17}$$

Hence $\bar{A}_\epsilon$ is Hurwitz for sufficiently small ϵ. Thus, from Lemma 6.1, an LCF of P_ϵ is given by $(\tilde{D}_\epsilon,\tilde{N}_\epsilon)$, where

$$\tilde{N}_\epsilon=C_0(sI-\bar{A}_\epsilon)^{-1}(B_\epsilon-F_0D_1)+D_1 \tag{6.18}$$

$$\tilde{D}_\epsilon=I-C_0(sI-\tilde{A}_\epsilon)^{-1}F_0 \tag{6.19}$$

with

$$B_\epsilon=\begin{bmatrix}B_1\\ B_2/\epsilon\end{bmatrix},\qquad C_0=[C_1 \quad C_2]. \tag{6.20}$$

The remainder of the proof consists of studying $\tilde{N}_\epsilon$ and $\tilde{D}_\epsilon$ in detail, and showing that $\tilde{N}_\epsilon\to\tilde{N}$, $\tilde{D}_\epsilon\to\tilde{D}$. Recall that the inverse of a partitioned matrix is given by

$$\begin{bmatrix}X & Y\\ W & V\end{bmatrix}^{-1}=\begin{bmatrix}\Delta^{-1} & -\Delta^{-1}YV^{-1}\\ -V^{-1}W\Delta^{-1} & V^{-1}+V^{-1}W\Delta^{-1}YV^{-1}\end{bmatrix} \tag{6.21}$$

where

$$\Delta=X-YV^{-1}W. \tag{6.22}$$

Hence,

$$\begin{aligned}\tilde{D}_\epsilon&=I-[C_1 \quad C_2]\begin{bmatrix}sI-A_{11}+FC_1 & -A_{12}+FC_2\\ -A_{21}/\epsilon & sI-A_{22}/\epsilon\end{bmatrix}^{-1}\begin{bmatrix}F\\0\end{bmatrix}\\ &=I-C_1\Delta^{-1}F+C_2V^{-1}W\Delta^{-1}F\end{aligned} \tag{6.23}$$

where

$$\begin{aligned}\Delta&=(sI-A_{11}+FC_1)-(A_{12}-FC_2)\left(sI-\frac{A_{22}}{\epsilon}\right)^{-1}\cdot\frac{A_{21}}{\epsilon}\\ &=(sI-A_{11}+FC_1)-(A_{12}-FC_2)(\epsilon sI-A_{22})^{-1}A_{21}\end{aligned} \tag{6.24}$$

$$V=sI-A_{22}/\epsilon,\qquad W=-A_{21}/\epsilon. \tag{6.25}$$

Simplification of (6.23) yields

$$\tilde{D}_\epsilon=I-M\Delta^{-1}F \tag{6.26}$$

where

$$M=C_1-C_2V^{-1}W=C_1+C_2(\epsilon sI-A_{22})^{-1}A_{21}. \tag{6.27}$$

Now

$$\begin{aligned}\tilde{D}&=I-C(sI-\bar{A})^{-1}F\\ &=I-C_1(sI-\bar{A})^{-1}F+C_2A_{22}^{-1}A_{21}(sI-\bar{A})^{-1}F.\end{aligned} \tag{6.28}$$

Hence, we can conclude that $\tilde{D}_\epsilon\to\tilde{D}$ if we can show that

$$\Delta^{-1}\to(sI-\bar{A})^{-1} \tag{6.29}$$

$$(\epsilon sI-A_{22})^{-1}A_{21}\Delta^{-1}\to -A_{22}^{-1}A_{21}(sI-\bar{A})^{-1}. \tag{6.30}$$

Note that $(\epsilon sI-A_{22})^{-1}$ does *not* approach $-A_{22}^{-1}$.[11] Hence, (6.30) does not automatically follow from (6.29).

To prove (6.29), let $\bar{A}_{11}$ denote $A_{11}-FC_1$, and note that

$$\begin{aligned}\Delta^{-1}&=\left[sI-\bar{A}_{11}-(A_{12}-FC_2)(\epsilon sI-A_{22})^{-1}A_{21}\right]^{-1}\\ &=\Big\{sI-\bar{A}_{11}-(A_{12}-FC_2)A_{22}^{-1}A_{21}\\ &\qquad-(A_{12}-FC_2)\left[(\epsilon sI-A_{22})^{-1}-A_{22}^{-1}\right]A_{21}\Big\}^{-1}\\ &=(I-R)^{-1}(sI-\bar{A})^{-1}\end{aligned} \tag{6.31}$$

where

$$R=(A_{12}-FC_2)\left[(\epsilon sI-A_{22})^{-1}-A_{22}^{-1}\right](sI-\bar{A})^{-1}. \tag{6.32}$$

Hence,

$$\Delta^{-1}-(sI-\bar{A})^{-1}=\left[(I-R)^{-1}-I\right](sI-\bar{A})^{-1}. \tag{6.33}$$

Now $(I-R)^{-1}(j\omega)$ approaches I uniformly on every *finite* interval $[-\omega_0,\omega_0]$ as $\epsilon\to 0$, and $(j\omega I-\bar{A})^{-1}\to 0$ as $|\omega|\to\infty$. So $\Delta^{-1}\to(sI-\bar{A})^{-1}$ and (6.29) is proved. To establish (6.30), observe that $(j\epsilon\omega I-A_{22})^{-1}$ approaches A_{22}^{-1} uniformly on every finite interval $[-\omega_0,-\omega_0]$ as $\epsilon\to 0$, and that both Δ^{-1}, $(sI-\bar{A})^{-1}$ are strictly proper. Thus, (6.30) follows from (6.29) plus the strict properness property of Δ^{-1} and its limit. Now (6.29) and (6.30) together show that $\tilde{D}_\epsilon\to D$ as $\epsilon\to 0$.

Now let us look at the "numerator" matrix $\tilde{N}_\epsilon$. From (6.18),

$$\tilde{N}_\epsilon=[C_1 \quad C_2]\begin{bmatrix}sI-A_{11}+FC_1 & -A_{12}+FC_2\\ -A_{21}/\epsilon & sI-A_{22}/\epsilon\end{bmatrix}^{-1}\cdot\begin{bmatrix}B_1-FD_1\\ B_2/\epsilon\end{bmatrix}+D_1. \tag{6.34}$$

[11] Observe that $(\epsilon s-A_{22})^{-1}\to 0$ as $|s|\to\infty$, whenever $\epsilon>0$. Hence, $\|(\epsilon s-A_{22})^{-1}-A_{22}^{-1}\|\geq\|A_{22}^{-1}\|$ no matter how small ϵ is.

Ignoring the D_1 term which is simply added on, we can write $\tilde{N}_\epsilon$ as a sum of four terms obtained by expanding (6.34). Using (6.21), the first term can be expressed as $C_1\Delta^{-1}(B_1 - FD_1)$. From (6.29), it follows that this converges to $C_1(sI - \bar{A})^{-1}(B_1 - FD_1)$. The second term is

$$\begin{aligned} &-C_2V^{-1}W\Delta^{-1}(B_1 - FD_1) \\ &= C_2\left(sI - \frac{A_{22}}{\epsilon}\right)^{-1}\frac{A_{21}}{\epsilon}\Delta^{-1}(B_1 - FD_1) \\ &= C_2(\epsilon sI - A_{22})^{-1}A_{21}\Delta^{-1}(B_1 - FD_1). \qquad (6.35) \end{aligned}$$

From reasoning analogous to that used to establish (6.30), it follows that this term converges to $-C_2A_{22}^{-1}A_{21}(sI - \bar{A})^{-1}(B_1 - FD_1)$. The third term is

$$\begin{aligned} -C_1\Delta^{-1}YV^{-1}B_2/\epsilon &= C_1\Delta^{-1}(A_{12} - F_1C_2)\cdot\left(sI - \frac{A_{22}}{\epsilon}\right)^{-1}\frac{B_2}{\epsilon} \\ &= C_1\Delta^{-1}(A_{12} - F_1C_2)(\epsilon sI - A_{22})^{-1}B_2. \end{aligned} \qquad (6.36)$$

As before, this converges to

$$\begin{aligned} &-C_1(sI - \bar{A})^{-1}(A_{12} - F_1C_2)A_{22}^{-1}B_2 \\ &\qquad = -C_1(sI - \bar{A})^{-1}A_{12}A_{22}^{-1}B_2 \qquad (6.37) \end{aligned}$$

since $C_2A_{22}^{-1}B_2 = 0$. The fourth and final term is

$$\begin{aligned} &C_2V^{-1}(I + W\Delta^{-1}YV^{-1})B_2/\epsilon \\ &= C_2\left(sI - \frac{A_{22}}{\epsilon}\right)^{-1} \\ &\quad\cdot\left[I + \frac{A_{21}}{\epsilon}\cdot\Delta^{-1}(A_{12} - FC_2)\left(sI - \frac{A_{22}}{\epsilon}\right)^{-1}\right]\cdot\frac{B_2}{\epsilon} \\ &= C_2(\epsilon sI - A_{22})^{-1}\left[I + A_{22}\Delta^{-1}(A_{12} - FC_2)(\epsilon sI - A_{22})^{-1}\right]B_2 \\ &= C_2(\epsilon sI - A_{22})^{-1}B_2 + C_2(\epsilon sI - A_{22})^{-1}A_{21}\Delta^{-1} \\ &\quad\cdot(A_{12} - FC_2)(\epsilon sI - A_{22})^{-1}B_2. \qquad (6.38) \end{aligned}$$

Now the second term on the right side of (6.38) is strictly proper and therefore converges without any difficulty to $C_2A_{22}^{-1}A_{21}(sI - \bar{A})^{-1}(A_{12} - FC_2)A_{22}^{-1}B_2$. Further, the first term is identically zero by assumption. Moreover, the limit of the second term simplifies to $C_2A_{22}^{-1}A_{21}(sI - \bar{A})^{-1}A_{12}A_{22}^{-1}B_2$. Putting everything together, we see that $\tilde{N}_\epsilon$ approaches

$$\begin{aligned} &(C_1 - C_2A_{22}^{-1}A_{21})(sI - \bar{A})^{-1}(B_1 - FD_1 - A_{12}A_{22}^{-1}B_2) + D_1 \\ &\qquad = C(sI - \bar{A})^{-1}(B - FD_1) + D_1 = \tilde{N} \qquad \text{since } D_1 = D. \end{aligned} \qquad (6.39)$$

Hence $P_\epsilon \to P_0$ in the graph topology. □

In conclusion, observe that if all matrices in (6.1), (6.2) are known, then Theorem 4.1 can be used to make the qualitative phrase "for small enough ϵ" more precise in a quantitative sense.

VII. Conclusions

In this paper, we have defined a "graph metric" that provides a measure of the distance between unstable plants. The graph metric induces a "graph topology" on unstable plants, which is the weakest possible topology in which feedback stability is robust. Using the graph metric, it is possible to derive estimates for the robustness of feedback stability *without* assuming that the perturbed and unperturbed plants have the same number of RHP poles. If the perturbed and unperturbed plants have the same RHP poles, then one can derive necessary and sufficient conditions for robustness with respect to a given class of perturbations.

The generalization of the results of this paper to linear distributed systems is straightforward. One simply replaces $\mathscr{S}$ by the set of transfer functions of all BIBO stable systems (lumped as well as distributed). This is the set $\hat{\mathscr{A}}$ defined in [18]. It is now known [7] that not all transfer functions of the form a/b, where $a, b \in \hat{\mathscr{A}}$, have coprime factorizations over $\hat{\mathscr{A}}$. Hence, the graph metric can only be defined for those plants that have an RCF over $\hat{\mathscr{A}}$. Fortunately, this class includes all lumped systems, as well as all the class $\text{mat}(\hat{B})$; see [11] and the references therein. The existence of the spectral factors needed to define the graph metric follows from [19]. In order to do full justice to the technicalities in the case of distributed systems, a detailed treatment will be given elsewhere.

In defining the graph metric between two plants, we have used their right-coprime factorizations. We could have also used their LCF's to define an equivalent, although not equal, metric.

There are some open problems for future research. The first is to find a formula for explicitly computing the graph metric (we give only an upper and a lower bound). The second is to improve the bounds given in Theorem 4.1 by making them less conservative.

Appendix
Comparison to the Gap Metric

In this Appendix, we analyze the gap metric defined in [14], [15] and analyze its relationship to the graph metric defined in the present paper. A few preliminaries are required for this purpose.

Recall [21] that the Hardy space H^2 consists of analytic functions f of the complex variable s with the property that

$$\sup_{\sigma > 0}\int_{-\infty}^{\infty}|f(\sigma + j\omega)|^2\,d\omega < \infty. \qquad (A1)$$

The space H^2 is a Hilbert space, with inner product defined by

$$\langle f, g\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty}f^*(j\omega)g(j\omega)\,d\omega. \qquad (A2)$$

Moreover, it is well known [21, p. 471] that a "time" function $f(t)$ belongs to $L_2[0,\infty)$ if and only if its Laplace transform $\hat{f}(s)$ belongs to H^2. Further, Laplace transformation is a linear isometry from L_2 into H^2.

Let $\mathscr{B}(L_2)$ denote the set of linear continuous operators mapping L_2 into itself. Then $\mathscr{B}(L_2)$ is a Banach algebra if each operator is equipped with the standard supremum norm. It is important to note that not all operators in $\mathscr{B}(L_2)$ are causal in the sense of [18, p. 39]. Since L_2 and H^2 are isomorphic, we let $\mathscr{B}(H^2)$ denote the set of continuous linear operators on H^2 corresponding to those in $\mathscr{B}(L_2)$.

The notion of stability employed in [14], [15] is that an input–output map is stable if it belongs to $\text{mat}(\mathscr{B}(L_2))$. Thus, a system is deemed to be stable if it maps L_2-inputs into L_2-outputs in a continuous (although not necessarily causal) manner. Thus, the above notion of stability is equivalent to requiring the system transfer matrix to lie in $\text{mat}(\mathscr{B}(H^2))$. This is a weaker notion of stability than the one employed in this paper, which (at this level of generality) corresponds to requiring the system transfer matrix to lie in $\text{mat}(\hat{\mathscr{A}})$ (see Section VII). Thus, we require a stable input–output map to be causal as well as bounded, whereas in [14], [15] a stable input–output map is only required to be bounded.

In [14], [15] a gap metric is defined on the set of closed operators (i.e., operators whose graphs are closed subspaces) mapping some subset of L_2^m into itself, for some integer m. If δ denotes the gap metric, then the principal result of [14], [15] can be stated as follows. Suppose $P \in \text{mat}(\mathscr{B}(L_2))$ is *square*, and that $\delta(P, P_1) < (1+\|P\|^2)^{-1/2}$. Then $P_1 \in \text{mat}(\mathscr{B}(L_2))$, and

$$\|P - P_1\| \leqslant \frac{(1+\|P\|^2)\delta(P,P_1)}{1-(1+\|P\|^2)^{1/2}\cdot\delta(P,P_1)}. \tag{A.3}$$

Even if P is causal, the above result does not guarantee that P_1 is *causal*, only that it is *bounded*. In fact causality is not addressed in [14], [15]. Using (A.3), it is shown that if $(I+P)^{-1} \in \text{mat}(\mathscr{B}(L_2))$ and $\delta(P, P_1)$ is sufficiently small, then $(I+P_1)^{-1} \in \text{mat}(\mathscr{B}(L_2))$, and $\|(I+P_1)^{-1}-(I+P)^{-1}\|$ is $0(\delta(P,P_1))$; the actual formulas can be found in [14], [15]. This result means that if P is stabilized by unit feedback and $\delta(P, P_1)$ is sufficiently small, then P_1 is also stabilized by unit feedback and $(I+P_1)^{-1}$ is close to $(I+P)^{-1}$.

Thus, in summary, a comparison of the contents of this paper with those of [14], [15] results in the following observations.

1) In the present paper, causality as well as boundedness of the input-output map are requirements for stability; in [14], [15] only boundedness is required. It is not known at present whether the results of [14], [15] can be modified to conclude the causality as well as boundedness of the perturbed system, if the original system is causal as well as bounded.

2) The present analysis is carried out for nonsquare plants and general (not necessarily unit or even stable) feedback. The analysis in [14], [15] is for square plants under unit feedback.

3) The scope of the results in [14], [15] can be extended by treating the plant–compensator combo as a square system. That is, given the system of Fig. 1, define

$$G = \begin{bmatrix} 0 & -C \\ P & 0 \end{bmatrix} \tag{A.4}$$

and observe that $H(P,C) = (I+G)^{-1}$. But in this case, $\delta(G, G_1)$ is not related in [14], [15] to individual variations in P or C. This is done in the present paper.

In [17], a topology is defined for unstable plants in a very general setting. One begins with a set $\mathscr{H}$ of "stable" plants, which is assumed to have two properties: i) $\mathscr{H}$ is a topological ring with no zero divisors, and ii) the set $\mathscr{U}$ of units in $\mathscr{H}$ is open, and the map $u \to u^{-1}$ mapping $\mathscr{U}$ into itself is continuous. The universe of unstable plants is then taken to be the set $\text{mat}(\mathscr{F})$, where $\mathscr{F}$ is the field of fractions associated with $\mathscr{H}$. Let $\mathscr{C}$ denote the subset of $\text{mat}(\mathscr{F})$ consisting of those matrices that have both an RCF as well as an LCF over $\mathscr{H}$; then a natural topology can be defined over $\mathscr{C}$ in a manner entirely analogous with Definition 2.1, by just replacing the set $\mathscr{S}$ by $\mathscr{H}$. We may refer to this as the *graph topology induced by* $\mathscr{H}$. In the remainder of this Appendix, we show that the topology induced by the gap metric is the same as the graph topology induced by the set $\mathscr{B}(L_2)$.

This comment in no way implies that the graph metric of Section III is equivalent to the gap metric. The topology of Section II (which is the one induced by the graph metric) is the graph topology induced by the set $\mathscr{S}$, whereas the topology induced by the gap metric is the graph topology induced by the set $\mathscr{B}(L_2)$. They are both graph topologies, corresponding to different choices for the set of "stable" systems. Since $\mathscr{S}$ (or $\hat{\mathscr{A}}$) is a *proper* subset of $\mathscr{B}(H^2)$, the topology on rational functions induced by the gap metric is certainly no stronger than the one induced by the graph metric. Whether it is actually weaker is not yet known.

Before presenting the main result of the Appendix, we state and prove a few lemmas. In what follows, P is always of dimension $m \times m$, and $\mathscr{C}$ denotes the set of plants that possess both an RCF and an LCF over $\mathscr{B}(L_2)$.

Lemma A.1: Suppose $P \in \mathscr{C}$. Then P is closed.

Proof: We will actually show that if P has an RCF, then the graph of P is closed. It will be seen that this result also holds for nonsquare plants. Let (N, D) be an RCF of P over $\mathscr{B}(L_2)$. Then, by an easy modification of [3, Theorem 2], it follows that the graph of P is described by

$$G(P) = \{(Dz, Nz) : z \in L_2^m\}. \tag{A.5}$$

Select X, Y in $\text{mat}(\mathscr{B}(L_2))$ such that $XN + YD = I$, and let $\{v_i\}$ be any sequence in $\mathscr{G}(P)$ converging to $v \in L_2^{2m}$; we will show that $v \in \mathscr{G}(P)$. Suppose $v_i = (Dz_i, Nz_i)$ where $z_i \in L_2^m$. Then $z_i = [Y \quad X]v_i$. Since $v_i \to v$, it follows that $\{z_i\}$ is convergent, with $[Y \quad X]v$ as its limit. Define $z = [Y \quad X]v \in L_2^m$. Then $(Dz, Nz) \in \mathscr{G}(P)$. Moreover, $(Dz, Nz) = \lim(Dz_i, Nz_i) = \lim v_i = v$. □

Since $\mathscr{G}(P)$ is a closed subspace of L_2^{2m}, there is a well-defined orthogonal projection mapping L_2^{2m} onto $\mathscr{G}(P)$. Let $\Pi(P)$ denote this projection. Then, for any $x \in L_2^{2m}$, $\Pi(P)x$ is the unique element v in $\mathscr{G}(P)$ that minimizes $\|x - v\|$. Since every $v \in \mathscr{G}(P)$ is of the form (Dz, Nz) for some $z \in L_2^m$, it is an easy exercise to show that

$$\Pi(P) = \begin{bmatrix} D \\ N \end{bmatrix} (D^*D + N^*N)^{-1} [D^* \quad N^*] \tag{A.6}$$

where * denotes the adjoint operator. Note that $\Pi(P)$ is self-adjoint, which means that it is noncausal except in the most trivial cases.

Lemma A.2: Suppose $(I+P)^{-1} \in \text{mat}(\mathscr{B}(L_2))$. Then $P \in \mathscr{C}$.

Proof: Let $(I+P)^{-1} = R$. Then $P = R^{-1} - I = (I-R)R^{-1} = R^{-1}(I-R)$. Since $R + (I-R) = I$, R and $I - R$ are both left- and right-coprime.

Lemma A.2 shows that $\mathscr{C}$ is large enough to include all plants stabilized by unit feedback.

Now we present the main result of this section. Note that the plants in Theorem A.1 need not be square.

Theorem A.1: A sequence $\{P_i\}$ in $\mathscr{C}$ converges to $P \in \mathscr{C}$ in the graph topology induced by $\mathscr{B}(L_2)$ if and only if $\delta(P, P_i) \to 0$.

Proof:

"Only If": Recall [15, p. 88] that

$$\delta(P, P_i) = \|\Pi(P) - \Pi(P_i)\|. \tag{A.7}$$

Suppose $P_i \to P$ in the graph topology induced by $\mathscr{B}(L_2)$. Then there exist RCF's (N_i, D_i) of P_i and (N, D) of P such that $D_i \to D$, $N_i \to N$ in $\mathscr{B}(L_2)$. Since formation of adjoints and inversion are both continuous on $\mathscr{B}(L2)$, it follows that

$$\Pi(P_i) = \begin{bmatrix} D_i \\ N_i \end{bmatrix} (D_i^*D_i + N_i^*N_i)^{-1} [D_i^* \quad N_i^*] \to \Pi(P). \tag{A.8}$$

"If": Suppose $\delta(P, P_i) \to 0$, and let (N, D) be an RCF of P over $\mathscr{B}(L_2)$. We will construct RCF's (N_i, D_i) of P_i such that $N_i \to N$, $D_i \to D$. In fact, define

$$\begin{bmatrix} D_i \\ N_i \end{bmatrix} = \Pi(P_i) \begin{bmatrix} D \\ N \end{bmatrix}. \tag{A.9}$$

Since $\Pi(P_i) \to \Pi(P)$,

$$\begin{bmatrix} D_i \\ N_i \end{bmatrix} \to \Pi(P) \begin{bmatrix} D \\ N \end{bmatrix} = \begin{bmatrix} D \\ N \end{bmatrix} \tag{A.10}$$

and all that remains to be shown is that (N_i, D_i) is actually an RCF of P_i. We state this as a separate lemma, since it might be of independent interest.

Lemma A.3: Suppose (N, D) is an RCF of P, that $\delta(P,P_1)<1$, and define (N_1, D_1) by

$$\begin{bmatrix} D_1 \\ N_1 \end{bmatrix} = \Pi(P_1)\begin{bmatrix} D \\ N \end{bmatrix}. \tag{A.10}$$

Then (N_1, D_1) is an RCF of P_1.

Proof: The proof is divided into four steps.

Step 1: The set $\{(D_1 z, N_1 z)\colon z \in L_2^m\}$ is a closed subspace of L_2^{2m}. We show that there exists a constant α such that

$$\|z\| \leqslant \alpha \left\| \begin{bmatrix} D_1 z \\ N_1 z \end{bmatrix} \right\| \qquad \text{for all } z \in L_2^m. \tag{A.11}$$

The above claim will then follow readily from [22, p. 513, Problem 15(ii)].

Let δ denote $\delta(P,P_1)$ and recall that $\delta<1$. Now, if $v \in \mathcal{G}(P)$, then

$$\|v - \Pi(P_1)v\| = \|\Pi(P)v - \Pi(P_1)v\| \leqslant \delta\|v\|. \tag{A.12}$$

Hence

$$\|\Pi(P_1)v\| \geqslant (1-\delta)\|v\| \qquad \text{for all } v \in \mathcal{G}(P). \tag{A.13}$$

So for any $z \in L_2^m$, we have

$$\left\| \begin{bmatrix} D_1 z \\ N_1 z \end{bmatrix} \right\| \geqslant (1-\delta) \left\| \begin{bmatrix} Dz \\ Nz \end{bmatrix} \right\|. \tag{A.14}$$

Select $X, Y \in \operatorname{mat}(\mathcal{B}(L_2))$ such that $XN + YD = I$. Then

$$z = [Y \quad X]\begin{bmatrix} Dz \\ Nz \end{bmatrix},$$

which implies that

$$\|z\| \leqslant \|[Y \quad X]\| \cdot \left\| \begin{bmatrix} Dz \\ Nz \end{bmatrix} \right\|. \tag{A.15}$$

The inequality (A.11) readily follows from (A.14) and (A.15).

Step 2: The set $\{(D_1 z, N_1 z)\colon z \in L_2^m\}$ equals $\mathcal{G}(P_1)$. In view of Step 1, it is enough to show that the range of $\begin{bmatrix} D_1 \\ N_1 \end{bmatrix}\colon L_2^m \to \mathcal{G}(P_1)$ is dense in $\mathcal{G}(P_1)$. Since $\mathcal{G}(P_1)$ is a closed subspace of L_2^{2m}, it is a Hilbert space in its own right. Hence, to establish the claim, it is enough to show that if $v \in \mathcal{G}(P_1)$ and $\left\langle v, \begin{bmatrix} D_1 \\ N_1 \end{bmatrix} z \right\rangle = 0$ for all $z \in L_2^m$, then $v = 0$. Suppose $v \in \mathcal{G}(P_1)$ has this property. Then

$$0 = \left\langle v, \begin{bmatrix} D_1 \\ N_1 \end{bmatrix} z \right\rangle = \left\langle [D_1^* \quad N_1^*]\, v, z \right\rangle \qquad \text{for all } z \in L_2^m \tag{A.16}$$

which implies that $[D_1^* \quad N_1^*]v = 0$, i.e., that $[D^* \quad N^*]\Pi(P_1)^* v = 0$ because of (A.10). Now $\Pi(P_1)$ is self-adjoint, and $v \in \mathcal{G}(P_1)$; so $\Pi(P_1)^* v = v$. Hence, $[D^* \quad N^*]v = 0$, which implies that $\pi(P)v = 0$, from (A.6). Now, interchanging P and P_1 in (A.13) gives

$$\|\Pi(P)v\| \geqslant (1-\delta)\|v\| \qquad \text{for all } v \in \mathcal{G}(P_1). \tag{A.17}$$

Since $\delta<1$, $\Pi(P)v = 0$ implies $v = 0$.

Step 3: The map $z \to (D_1 z, N_1 z)$ is a one-to-one map of L_2^m onto $\mathcal{G}(P_1)$. The "onto" part is established in Step 2, and the one-to-one part follows from (A.11).

Step 4: Let $(\overline{N}_1, \overline{D}_1)$ be any RCF of P_1, and select $X_1, Y_1 \in \operatorname{mat}(\mathcal{B}(L_2))$ such that $X_1\overline{N}_1 + Y_1\overline{D}_1 = I$; then $X_1 N_1 + Y_1 D_1$ is a unit of $\operatorname{mat}(\mathcal{B}(L_2))$. Clearly this step completes the proof. Let $U = X_1 N_1 + Y_1 D_1$. The action of U can be simply explained as follows. Every $v \in \mathcal{G}(P_1)$ can be uniquely represented as $(D_1 z, N_1 z)$ for some $z \in L_2^m$, and as $(\overline{D}_1 \bar{z}, \overline{N}_1 \bar{z})$ for some $\bar{z} \in L_2^m$, U merely maps z into $\bar{z}$. As such, U is a continuous one-to-one map of L_2^m onto itself, and, by the open mapping theorem [22, p. 57], U^{-1} is also continuous. □

In closing, we observe that all of the above reasoning breaks down completely if we insist on using *causal* bounded operators instead of $\mathcal{B}(L_2)$.

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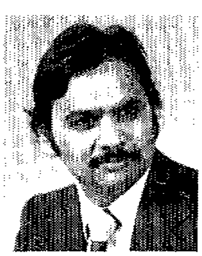

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On the Structure of State-Space Models for Discrete-Time Stochastic Vector Processes

ANDERS LINDQUIST AND MICHELE PAVON

Abstract—**From a conceptual point of view, structural properties of linear stochastic systems are best understood in a geometric formulation which factors out the effects of the choice of coordinates. In this paper we study the structure of discrete-time linear systems with stationary inputs in the geometric framework of splitting subspaces set up in the work by Lindquist and Picci. In addition to modifying some of the realization results of this work to the discrete-time setting, we consider some problems which are unique to the discrete-time setting. These include the relations between models with and without noise in the observation channel, and certain degeneracies which do not occur in the continuous-time case. One type of degeneracy is related to the singularity of the state transition matrix, another to the rank of the observation noise and invariant directions of the matrix Riccati equation of Kalman filtering. We determine to what extent these degeneracies are properties of the output process. The geometric framework also accommodates infinite-dimensional state spaces, and therefore the analysis is not limited to finite-dimensional systems.**

I. Introduction

THIS paper is concerned with stochastic realization of discrete-time stationary vector processes and the structural properties of the resulting stochastic systems. Although our results provide new insight into the finite-dimensional case, the analysis is not restricted to finite-dimensional systems. The significance of a state-space theory for infinite-dimensional systems has been stressed by many authors in the deterministic context [1]–[4].

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The stochastic realization problem is the centerpiece of any theory of stochastic systems. The early results in this field of study were developed in the context of spectral factorization and the positive-real lemma [5]–[8]. In more recent years, however, there has been a trend toward a more geometric approach [14]–[39]. This has several advantages from a conceptual point of view. First, there is no need to restrict the analysis to finite-dimensional systems: the geometric properties are in general (but not always) independent of dimension. Second, it allows us to factor out, in the first analysis, the properties of realizations which depend only on the choice of coordinates. In fact, the geometric approach is coordinate-free. Structural properties which look very complicated in their coordinate-dependent form are given geometric descriptions. Third, systems-theoretical concepts such as minimality, observability, constructibility, etc., can be defined and analyzed in geometric terms. We hasten to stress, however, that such theory does not *replace* the classical results. Indeed, we shall still need to do spectral factorization. The emphasis in the geometric approach is on the structural aspects of the problem rather than on the algorithmic ones, although the insights gained by this analysis may be helpful in providing better algorithms.

In this paper we use the geometric format laid out by Lindquist and Picci [19]–[24] to develop a theory of stochastic realization for discrete-time processes. Since much of the basic geometry is the same in continuous and discrete time, and hence is covered in [19]–[24], our emphasis here is on structural properties which are unique to the discrete-time setting, and which have not been covered elsewhere (such as in the work by Ruckebusch [28]–[32], which deals mainly with the discrete-time case). In addition to working out the details on difference-equation representations, we consider questions concerning the manner in which noise enters into the observation channel and the relations between models with and without observation noise. We study the types of degeneracy which manifest themselves either by the transition function being singular or the observation noise being deficient in rank. The first type of degeneracy occurs in the important class of moving-average processes, whereas the second one is related to