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A GENERALIZATION OF THE RELATIVISTIC THEORY OF GRAVITATION, II

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In a previous paper (Ann. of Math., Vol. 46, No. 4) one of us developed a generally relativistic theory, which is characterized as follows:

(1) Group of real transformations of the four coordinates $(x_1, \cdots, x_4)$

(2) As only dependent variable to which everything is reduced we have the tensor g_{ik}, which is taken there to be complex and of Hermitian symmetry. W. Pauli noted, that the theory developed on this basis is such that the limitation to the case of the Hermitian tensor is not needed for the formalism.

(3) It was added in proof that it seems natural to assume that the field satisfy the equations

$$\Gamma_i = \tfrac{1}{2}(\Gamma^a_{ia} - \Gamma^a_{ai}) = 0. \tag{1}$$

It was asserted but not proven, that there exist identities which allow us to adjoin these equations without introducing an impermissible overdetermination. This assertion was, however, based on an error. The introduction of equation (1) implies a different derivation of the field equations from the original one and a (slight) deviation of the latter from the field equations of the first paper.

The mathematical formalism of the theory is preserved here except for an alteration relative to the rules for absolute differentiation of tensor densities. Otherwise knowledge of that formalism is assumed here.

§1. The dependence of the infinitesimal parallel translation of the fundamental tensor. Absolute differentiation of densities.

The connection between the g_{ik} and the Γ^a_{ik} is characteristic for the theory. It is given by the equation:

$$(g_{\underset{+-}{ik};a} \equiv) g_{ik,a} - g_{sk}\Gamma^s_{ia} - g_{is}\Gamma^s_{ak} = 0. \tag{2}$$

This determination of the Γ from the g has the following property:
If to the tensor g_{ik} corresponds the translation Γ^a_{ik} according to (2), then to the tensor $\tilde{g}_{ik} = g_{ki}$ corresponds $\tilde{\Gamma}^a_{ik} = \Gamma^a_{ki}$.

PROOF: If one forms the left side of (2) for the $\tilde{g}_{ik}$ and the $\tilde{\Gamma}^a_{ik}$ one gets

$$\tilde{g}_{ik,a} - \tilde{g}_{sk}\tilde{\Gamma}^s_{ia} - \tilde{g}_{is}\tilde{\Gamma}^s_{ak};$$

if we introduce here the g and Γ, according to the above definition, and exchange the last two terms we get

$$g_{ki,a} - g_{si}\Gamma^s_{ka} - g_{ks}\Gamma^s_{ai}.$$

This expression vanishes according to our assumption, since it becomes the left side of equation (2) if we interchange the free indices i and k.

REMARK: The property just established has nothing to do with the assumption that g_{ik} and Γ^a_{ik} are Hermitian with respect to the indices i and k. It is as possible and as natural to consider these quantities to be real but not symmetric; the number of independent components of g and Γ is then the same as in the case of Hermitian symmetry. One thus obtains a theory which differs from the previously developed one by the signs of certain terms only.

Absolute differentiation of tensor densities

If we multiply the left-hand side of (2) by $\frac{1}{2}g^{ik}$ we get (see $\cdots$loc. cit) the vector

$$\frac{(\sqrt{-g})_{,\alpha}}{\sqrt{-g}} - \tfrac{1}{2}(\Gamma^s_{as} + \Gamma^s_{sa}); \tag{2.1}$$

multiplying by $\sqrt{-g}$ we get the vector density

$$(\sqrt{-g})_{,\alpha} - \tfrac{1}{2}\sqrt{-g}(\Gamma^s_{as} + \Gamma^s_{sa}).$$

This we define as the absolute derivative $(\sqrt{-g})_{,\alpha}$ of the scalor density $\sqrt{-g}$. Correspondingly we define the absolute derivative of every scalor density ρ

$$\rho_{;a} = \rho_{,a} - \rho\,\tfrac{1}{2}(\Gamma^s_{as} + \Gamma^s_{sa}). \tag{3}$$

From this the rules of differentiation for all tensor densities follow in a well known manner, e.g.

$$\mathfrak{g}^{ik}_{+-;a} = \mathfrak{g}^{ik}{}_{,a} + \mathfrak{g}^{sk}\Gamma^i_{sa} + \mathfrak{g}^{is}\Gamma^k_{as} - \mathfrak{g}^{ik}\tfrac{1}{2}(\Gamma^s_{as} + \Gamma^s_{sa}). \tag{3.1}$$

It can be easily shown that the equations

$$g^{i\,k}_{+-};\,l = 0;\ g^{ik}_{+-}\ ;\,l = 0;\ \mathfrak{g}^{i\,k}_{+-}\ ;\,l = 0$$

are equivalent here too.

When (2) is satisfied, then the rule of differentiation for tensor densities corresponds to the one defined previously.

For a contravariant vector density $\mathfrak{A}^i$ we get

$$\mathfrak{A}^i_{+;a} = \mathfrak{A}^i{}_{,a} + \mathfrak{A}^s\Gamma^i_{sa} - \mathfrak{A}^i\tfrac{1}{2}(\Gamma^s_{as} + \Gamma^s_{sa})$$

and for the divergence

$$\mathfrak{A}^a_{+;a} = \mathfrak{A}^a{}_{,a} + \mathfrak{A}^a\Gamma_a, \tag{3.2}$$

also

$$\mathfrak{A}^a_{-;a} = \mathfrak{A}^a{}_{,a} - \mathfrak{A}^a\Gamma_a. \tag{3.3}$$

Here we see how natural it is to specialize the field by equation (1). For on the right-hand sides of (3.2) and (3.3) each term has tensor character, but according to (1) there will be only one term.

There are other formal reasons for postulating equations (1) which we should mention here. Like in the theory of symmetrical g_{ik} the once contracted curvature tensor plays an important part. The curvature tensor

$$R^i_{klm} \equiv \Gamma^i_{kl,m} - \Gamma^i_{al}\Gamma^a_{km} - \Gamma^i_{km,l} + \Gamma^i_{am}\Gamma^a_{kl}$$

has a contraction with respect to the indices i and k which vanishes identically in the original theory of gravitation.

Here we get

$$R^a_{alm} = \Gamma^a_{al,m} - \Gamma^a_{am,l}$$

which in general does not vanish even if (2) is satisfied. Namely, if we transform the right-hand side using the equation following from (2.1)

$$(2.2) \qquad (\Gamma^a_{al} + \Gamma^a_{la})_{,m} - (\Gamma^a_{am} + \Gamma^a_{ma})_{,l} \equiv 0$$

we get

$$R^a_{a\,lm} \equiv -(\Gamma_{l,m} - \Gamma_{m,l}).$$

This will not vanish in general, but, it will vanish when the field satisfies equation (1).

If we contract $R^i_{k\,lm}$ according to the indices i and m, we get the tensor

$$R_{kl} \equiv R^a_{kla} \equiv \Gamma^a_{kl,a} - \Gamma^a_{kb}\Gamma^b_{al} - \Gamma^a_{ka,l} + \Gamma^a_{kl}\Gamma^b_{ab}.$$

This tensor is, in general, not Hermitian, i.e., it is not transformed into itself if we replace the Γ by the $\tilde{\Gamma}$ and interchange the indices k and l. (In the following we shall use the terminology Hermitian in this sense.) For the anti-Hermitian part we get:

$$2R^*_{kl} = -\Gamma^a_{ka,l} + \Gamma^a_{al,k} + 2\Gamma^a_{kl}\Gamma_a;$$

considering (2.2) this becomes

$$R^*_{kl} = -\tfrac{1}{2}(\Gamma_{\underset{+}{k};l} + \Gamma_{\underset{-}{l};k}),$$

hence the anti-Hermitian part of R_{kl} vanishes when (1) and (2) are satisfied.

It would be easy to give further arguments to show that equation (1) is suitable for the space structure used. However, the above should suffice. It is now our task to find compatible field equations (on the basis of a variational principle) so that equations (1) and (2) are part of the field equations.

First we want to make another formal remark, which serves to prepare the derivation of the field equations. If in (3.1) we contract to form $\mathfrak{g}^{\underset{+}{i}\underset{-}{a}}{}_{;a}$ and $\mathfrak{g}^{\underset{+}{a}\underset{-}{i}}{}_{;a}$ then by subtraction we get

$$(3.4) \qquad \tfrac{1}{2}(\mathfrak{g}^{\underset{+}{i}\underset{-}{a}}{}_{;a} - \mathfrak{g}^{\underset{+}{a}\underset{-}{i}}{}_{;a}) \equiv \mathfrak{g}^{ia}_{\vee}{}_{,a} - \mathfrak{g}^{ia}_{-}\Gamma_a$$

where $\mathfrak{g}^{ia}_{-}$ is the symmetric, $\mathfrak{g}^{ia}_{\vee}$ the anti-symmetric part of $\mathfrak{g}^{ia}$. Hence, if (2) is satisfied we have identically

$$(3.5) \qquad (\mathfrak{g}^{ia}_{-}\Gamma_a)_{,i} \equiv 0$$

The equations (1) satisfy, therefore, a scalar identity as a result of (2). From equation (3.4) we see that equations (1) and (2) imply

$$(3.6) \qquad \mathfrak{g}^{ia}_{\vee;\,a} = 0.$$

§2. Hamiltonian. Field equations

We now choose the Hamiltonian

$$\mathfrak{H} = \mathfrak{g}^{ik}P_{ik} + \mathfrak{A}^i\Gamma_i + b_i\mathfrak{g}^{ia}_{\vee}{}_{,a}.$$

P_{ik} is the Hermitianized curvature tensor

$$P_{ik} = \Gamma^a_{ik,a} - \tfrac{1}{2}(\Gamma^a_{ia,k} + \Gamma^a_{ak,i}) - \Gamma^a_{ib}\Gamma^b_{ak} + \Gamma^a_{ik}\Gamma^b_{a\underset{-}{b}}.$$

The variation is performed according to the variables $\mathfrak{g}^{ik}$, Γ^a_{ik}, $\mathfrak{A}^i$, b_i which play the role of independent field variables, where the latter two (purely imaginary) quantities play the role of Lagrange multipliers. (Neither (1) nor (2) are assumed satisfied a priori.)

The variation according to the $\mathfrak{A}^i$ and b_i yields the equations

$$\Gamma_i = 0 \tag{4}$$

$$\mathfrak{g}^{ia}_{\vee}{}_{,a} = 0. \tag{5}$$

For the variation according to the Γ we use the method which has been established by Palatini for the case of symmetric g and Γ. It is easy to verify that

$$\delta P_{ik} = (\delta\Gamma^{\;\;a}_{\underset{+-}{ik}\;0})_{;a} - \tfrac{1}{2}(\delta\Gamma^a_{\underset{+}{i}a})_{;k} - \tfrac{1}{2}(\delta\Gamma^a_{a\underset{-}{k}})_{;i}$$

considering this the variation of the $\mathfrak{H}$-integral according to Γ (for $\delta\Gamma$ which vanish at the boundaries of integration).

$$(6)\quad \begin{cases} 0 = -\mathfrak{g}^{\underset{+-}{ik}}{}_{;a} + \tfrac{1}{2}\mathfrak{g}^{\underset{+-}{is}}{}_{;s}\delta^k_a + \tfrac{1}{2}\mathfrak{g}^{\underset{+-}{sk}}{}_{;s}\delta^i_a \\ \quad + \tfrac{1}{2}\mathfrak{g}^{is}\Gamma_s\delta^k_a - \tfrac{1}{2}\mathfrak{g}^{sk}\Gamma_s\delta^i_a \\ \quad + \tfrac{1}{2}\mathfrak{A}^i\delta^k_a - \tfrac{1}{2}\mathfrak{A}^k\delta^i_a. \end{cases}$$

The second line of (6) vanishes because of (4). If we contract (6) first according to k and a, then according to i and a we get the two equations

$$(6.1)\quad \begin{cases} \mathfrak{g}^{\underset{+-}{is}}{}_{;s} + \tfrac{1}{2}\mathfrak{g}^{\underset{+-}{si}}{}_{;s} + \tfrac{3}{2}\mathfrak{A}^i = 0 \\ \mathfrak{g}^{\underset{+-}{si}}{}_{;s} + \tfrac{1}{2}\mathfrak{g}^{\underset{+-}{is}}{}_{;s} - \tfrac{3}{2}\mathfrak{A}^i = 0. \end{cases}$$

Adding these two equations we get

$$\mathfrak{g}^{\underset{+-}{is}}{}_{;s} + \mathfrak{g}^{\underset{+-}{si}}{}_{;s} = 0. \tag{6.2}$$

Equation (3.4) which was based on the definition of absolute differentiation yields considering (4) and (5)

$$\mathfrak{g}^{\underset{+-}{is}}{}_{;s} - \mathfrak{g}^{\underset{+-}{si}}{}_{;s} = 0. \tag{3.7}$$

Hence $\mathfrak{g}^{\underset{+-}{is}}{}_{;s}$ and $\mathfrak{g}^{\underset{+-}{si}}{}_{;s}$ vanish and therefore (6.1) implies that $\mathfrak{A}^i$ vanishes. Equation (6) reduces therefore to

$$\mathfrak{g}^{\underset{+-}{ik}}{}_{;a} = 0. \tag{6.3}$$

Equation (5) is implied by equations (4) and (6.3) according to (3.4).

The variation of the $\mathfrak{H}$-integral according to the $\mathfrak{g}^{ik}$ yields

(7) $$P_{ik} - \tfrac{1}{2}(b_{i,k} - b_{k,i}) = 0$$

or separating according to symmetry

(7.1) $$P_{\underline{ik}} = 0$$

(7.2) $$P_{\underset{\vee}{ik}} - \tfrac{1}{2}(b_{i,k} - b_{k,i}) = 0$$

or, after elimination of the auxiliary variables b

(7.3) $$P_{\underset{\vee}{ik},l} + P_{\underset{\vee}{kl},i} + P_{\underset{\vee}{li},k} = 0.$$

Compiling the results of the variation, we get the field equations (which deviate slightly from (15b) of the first paper)

(8.1) $$\mathfrak{g}^{\underset{+\,-}{ik}}{}_{;a} = 0$$

(8.2) $$\Gamma_i = 0$$

(8.3) $$P_{\underline{ik}} = 0$$

(8.4) $$P_{\underset{\vee}{ik},l} + P_{\underset{\vee}{kl},i} + P_{\underset{\vee}{li},k} = 0$$

The derivation of these equations from a variational principle (with real $\mathfrak{H}$) guarantees their compatibility sufficiently.

If we compare the system of equations with that of the previous paper, we realize that equation (8.2) is introduced at the cost of weakening the equations which are derived from the curvature. Of the equations (8.4) only three are independent, while in the original formulation of the theory it corresponded to six equations; in addition the order of differentiation of the last equation has been raised by one. The introduction of the last term in the Hamiltonian, which caused this raise of the degree of differentiation, is necessary in order that (8.1) will hold, which is obviously the only reasonable determination of the Γ from the g.

Considering equations (8) the question arises, whether (8.3) and (8.4) could not be replaced by the stronger equation.

(9) $$P_{ik} = 0.$$

The question of the justification of such an equation caused us considerable trouble. This equation would obviously be justified if the equations (9), (8.1) and (8.2) would satisfy 3 independent additional identities. The assumption of the existence of such identities is strengthened by the fact that for infinitely weak fields such additional identities do indeed exist.

Namely, if we put (neglecting the special character of time)

$$g_{ik} = \delta_{ik} + \gamma_{ik}$$

and neglect the square of γ as compared to 1, then we may replace equations (8.1), (8.2) and (9) by the linearized ones

$$\gamma_{ik,a} - \Gamma^k_{ia} - \Gamma^i_{ak} = 0$$

$$\tfrac{1}{2}(\Gamma^s_{is} - \Gamma^s_{si}) = 0$$

$$\Gamma^s_{ik,s} - \tfrac{1}{2}\Gamma^s_{is,k} - \tfrac{1}{2}\Gamma^s_{sk,i} = 0.$$

From the first equation we solve for Γ

$$\Gamma^a_{ik} = \tfrac{1}{2}(-\gamma_{ki,a} + \gamma_{ia,k} + \gamma_{ak,i})$$

the second equation then gives

$$(G_i \equiv)\gamma_{\underset{\vee}{ia},a} = 0$$

and the third, considering this

$$(G_{ik} \equiv) - \gamma_{ki,aa} + \gamma_{ia,ak} + \gamma_{ak,ai} - \gamma_{aa,ik} = 0.$$

The latter antisymmetrized can be replaced considering $G_i = 0$ by

$$(U_{\underset{\vee}{ik}} \equiv)\ \gamma_{\underset{\vee}{ik},aa} = 0.$$

We now have the identity

$$U_{\underset{\vee}{ik},k} - G_{i,kk} \equiv 0.$$

If to this identity of the equations for the infinitely weak fields, there would correspond an identity of the rigorous equations, then the introduction of the stronger equation (9) would be justified. A complicated systematic investigation has shown that no such rigorous identity exists.

One may ask, if not despite the absence of these identities, the introduction of equation (9) may be considered. This, too, has to be answered in the negative on the basis of a consideration which is applicable also in other cases.

Let us assume that we have a system of equations $G = 0$ for which there exists a *rigorous* identity, which is linear and homogeneous in the equations. Written symbolically

$$L(G) \equiv 0$$

where L is an operator which is linear and homogeneous in the G. Now L and G can be developed according to the powers of the field quantities and their derivatives.

$$(L_0 + L_1 + \cdots)(G_1 + G_2 + \cdots) \equiv 0$$

whereby the identity divides according to powers of the field quantities. The first two are

$$L_0(G_1) \equiv 0$$

$$L_0(G_2) + L_1(G_1) \equiv 0.$$

We now assume that we have a parameter solution for G of the field quantities g

$$G(\epsilon g_1 + \epsilon^2 g_2 + \cdots) = 0$$

or

$$(G_1 + G_2 + \cdots)(\epsilon g_1 + \epsilon^2 g_2 + \cdots) = 0$$

or

$$G_1(\epsilon g_1 + \epsilon^2 g_2 + \cdots) + G_2(\epsilon g_1 + \epsilon^2 g_2 + \cdots) + \cdots = 0.$$

This shall be identically satisfied in ϵ. This yields the first two equations.

$$G_1(\epsilon g_1) = 0 \text{ or } G_1(g_1) = 0 \text{ (linear in } g)$$

$$G_1(\epsilon^2 g_2) + G_2(\epsilon g_1) = 0 \text{ or } G_1(g_2) + G_2(g_1) = 0.$$

We now apply our identity. Since we had $L_0(G_1) \equiv 0$ we get, applying L_0 to the second equation

(a) $$L_0(G_2(g_1)) = 0.$$

This is an equation of the second degree in g. Since we saw above that

(b) $$L_0(G_2) + L_1(G_1) \equiv 0$$

we know that this quadratic equation is a result of the linear equations

$$G_1(g_1) = 0.$$

If a linear identity exists for the first approximation to which there corresponds no rigorous identity, (as in the case of our equations) then we derive equations (a) as before but since the identity (b) will not hold in general this equation is no longer a result of the linearized field equations $G_1(g_1) = 0$. They are therefore additional equations for the first approximation. In the case of the field equations under our consideration they are so constructed that each of their terms is a product of a symmetric by an antisymmetric g (γ_{ik}) (or derivatives of these quantities).

If we interpret the symmetric $\underline{\gamma}_{ik}$ as an expression of the gravitation field and the $\underset{\vee}{\gamma}_{ik}$ as an expression of the electromagnetic field, then for the first approximation of the field we get a dependence of the electric of the gravitation field which cannot be brought in accord with our physical knowledge, therefore the considered strengthenng of equations (8) is out of the question.

The linearized equations which according to (8) hold for an antisymmetric (electromagnetic) field are

$$\underset{\vee}{\gamma}_{ik,k} = 0$$

$$(\underset{\vee}{\gamma}_{ik,l} + \underset{\vee}{\gamma}_{kl,i} + \underset{\vee}{\gamma}_{li,k})_{,ss} = 0$$

If, in the second equation, the expression inside the parentheses would itself vanish, then we would have Maxwell's equations for empty space, whose solutions therefore satisfy our equations. The latter seem to be too weak. This, however, is not a (justified) objection to the theory since we do not know to which solutions of the linearized equations there correspond *rigorous* solutions which are regular in the entire space. It is clear from the start that in a consistent field theory which claims to be complete (in contrast e.g. to the pure theory of gravitation) only those solutions are to be considered which are regular in the entire space. Whether such (non-trivial) solutions exist is as yet unknown.

§3. Conditions for the g_{ik} which follow from equation (2)

We now wish to investigate what conditions the g_{ik} have to satisfy in order that equations (2) determine the Γ uniquely and without singularities.

In the following we write: $g_{ik} = s_{ik}$; $g_{ik} = a_{ik}$. At each point we can transform the coordinates so that $s_{ik} = s_i\delta_{ik}$ (i not an index of summation). Equation (2) becomes:

(2.3) $a_{sk}\Gamma^s_{ia} + a_{is}\Gamma^s_{ak} + s_k\Gamma^k_{ia} + s_i\Gamma^i_{ak} = g_{ik,a}$ (i, k not indices of summation).

If we let $i = a = k$ we get

$$2s_i\Gamma^i_{ii} = g_{ii,i}. \tag{2.4}$$

Hence we must have

$$s_i \neq 0 \tag{10}$$

or, in other words, at every point we have for the determinant $| s_{ik} |$

$$| s_{ik} | \neq 0. \tag{10.1}$$

This result is important since it implies the existence of a "light cone" whose signature is the same everywhere. The division of the line elements into spacelike and timelike elements is thereby secured.

If the signature is now chosen as is customary in the theory of relativity, we can specialize the coordinates further so that

$$s_{ik} = s\eta_{ik}\left(s > 0 \quad \text{and} \quad \eta_{ik} = \begin{cases} \delta_{ik}\, i = 1, 2, 3 \\ -\delta_{ik} \text{ for } i = 4 \end{cases}\right).$$

We can also perform a (local) Lorentz transformation so that at our point all a_{ik} except $a_{12} = -a_{21}$ and $a_{34} = -a_{43}$ vanish. We write $a_{12} = a_1\epsilon_{12}$; $a_{34} = a_2\epsilon_{34}$. In the following, we shall use capital Roman indices for 1, 2, and Greek indices for 3, 4.

Consider first the equations

$$a_1(\epsilon_{SK}\Gamma^S_{IA} + \epsilon_{IS}\Gamma^S_{AK}) + s(\Gamma^K_{IA} + \Gamma^I_{AK}) = g_{IK,A}\, . \tag{2.5}$$

Where all indices are 1, 2 and not all A, I, K are equal. We then get six equations in six unknowns (ignoring Γ^A_{AA} which we got from (2.2)) with the determinant

$\Gamma =$	Γ^1_{12}	Γ^1_{21}	Γ^1_{22}	Γ^2_{11}	Γ^2_{12}	Γ^2_{21}
$(A, I, K) =$ (1, 1, 2)	s	0	0	s	a_1	0
(1, 2, 1)	0	s	0	s	0	$-a_1$
(1, 2, 2)	$-a_1$	a_1	0	0	s	s
(2, 1, 1)	s	s	0	0	$-a_1$	a_1
(2, 1, 2)	a_1	0	s	0	s	0
(2, 2, 1)	0	$-a_1$	s	0	0	s

$$= -4s^2(s^2 + a_1^2)^2.$$

In an analogous manner the determinant of the equations

(2.6) $$a_2(e_{\sigma K}\Gamma^{\sigma}_{\iota\alpha} + e_{\iota\sigma}\Gamma^{\sigma}_{\alpha\kappa}) + s(\eta_{\sigma\kappa}\Gamma^{\sigma}_{\iota\alpha} + \eta_{\sigma\iota}\Gamma^{\sigma}_{\alpha\kappa}) = g_{\iota\kappa,\alpha}$$

is

$$4s^2(-s^2 + a_2^2)^2.$$

Hence:

(11) $$(s^2 + a_1^2)(-s^2 + a_2^2) \neq 0$$

or

(11.1) $$g = |g_{ik}| \neq 0.$$

We now know that the g^{ik} exist (a fact which we had tacitly assumed before). We have in fact:

(12) $$g^{IK} = \frac{1}{s^2 + a_1^2}(s\delta_{IK} + a_1 e_{IK}); \quad g^{\iota\kappa} = \frac{1}{-s^2 + a_2^2}(-s\eta_{\iota\kappa} + a_2 \epsilon_{\iota\kappa}).$$

Then, if in the three equations:

$$g_{sk}\Gamma^s_{il} + g_{is}\Gamma^s_{lk} = g_{ik,l}$$

$$g_{sl}\Gamma^s_{ki} + g_{ks}\Gamma^s_{il} = g_{kl,i}$$

$$g_{si}\Gamma^s_{lk} + g_{ls}\Gamma^s_{ki} = g_{li,k}$$

we multiply the first by $g^{ml}g^{ak}$ the second by $-g^{ka}g^{ml}$ and the third by $g^{lm}g^{ka}$ and add, we get:

(2.7) $$(g^{ak}g^{ml}g_{is} + g^{ka}g^{lm}g_{si})\Gamma^s_{lk} = g^{ml}g^{ak}g_{ik,l} + g^{lm}g^{ka}g_{li,k} - g^{ml}g^{ka}g_{kl,i}.$$

Let us first consider the case

$$s = \sigma; l = L; k = K; i = \iota; m = M; a = A$$

using (12) the left-side of (2.7) becomes

(2.8) $$\frac{2s}{(s^2 + a_1^2)^2}[(s^2\delta_{AK}\delta_{ML} + a_1^2 e_{AK} e_{ML})\eta_{\iota\sigma} + a_1 a_2(\delta_{AK}\epsilon_{ML} + \delta_{ML} e_{AK})e_{\iota\sigma}]\Gamma^{\sigma}_{LK}.$$

The determinant is

$(i, A, M) =$ \ $\Gamma =$	Γ^3_{11}	Γ^3_{12}	Γ^3_{21}	Γ^3_{22}	Γ^4_{11}	Γ^4_{12}	Γ^4_{21}	Γ^4_{22}
(3, 1, 1)	s^2	0	0	a_1^2	0	a_1a_2	a_1a_2	0
(3, 1, 2)	0	s^2	$-a_1^2$	0	$-a_1a_2$	0	0	a_1a_2
(3, 2, 1)	0	$-a_1^2$	s_2	0	$-a_1a_2$	0	0	a_1a_2
(3, 2, 2)	a_1^2	0	0	s_2	0	$-a_1a_2$	$-a_1a_2$	0
(4, 1, 1)	0	$-a_1a_2$	$-a_1a_2$	0	$-s^2$	0	0	$-a_1^2$
(4, 1, 2)	a_1a_2	0	0	$-a_1a_2$	0	$-s^2$	a_1^2	0
(4, 2, 1)	a_1a_2	0	0	$-a_1a_2$	0	a_1^2	$-s^2$	0
(4, 2, 2)	0	a_1a_2	a_1a_2	0	$-a_1^2$	0	0	$-s^2$

$$\cdot \frac{(2s)^8}{(s^2 + a_1^2)^{16}} = \begin{vmatrix} s^2 & a_1^2 & a_1 a_2 & a_1 a_2 \\ a_1^2 & s^2 & -a_1 a_2 & -a_1 a_2 \\ a_1 a_2 & -a_1 a^2 & -s^2 & a_1^2 \\ a_1 a_2 & -a_1 a_2 & a_1^2 & -s_2 \end{vmatrix}^2 \cdot \frac{(2s)^8}{(s^2 + a_1^2)^{16}} =$$

$$\frac{(2s)^8}{(s^2 + a_1^2)^{12}} [(s^2 - a_1^2)^2 + 4a_1^2 a_2^2]^2$$

which gives us the condition

$$(s^2 - a_1^2)^2 + 4a_1^2 a_2^2 \neq 0 \tag{13}$$

or, in other words, we cannot at the same time have

$$a_1 = \pm s \quad \text{and} \quad a_2 = 0.$$

We see immediately that the equations for $\Gamma^S_{\lambda K}$ and $\Gamma^S_{L\kappa}$ have the determinants

$$\frac{(2s)^8}{(s^2 + a_1^2)^4(-s^2 + a_2^2)^8} [(s^2 - a_1^2)^2 + 4a_1^2 a_2^2]^2$$

and therefore yield no new inequalities.

In an analogous manner the equations of $\Gamma^S_{\lambda\kappa}$ have the determinant

$$\frac{(2s)^8}{(-s^2 + a_2^2)^{12}} [(s^2 + a_2^2)^2 + 4a_1^2 a_2^2]^2$$

which gives us the condition

$$(s^2 + a_2^2)^2 + 4a_1^2a_2^2 \neq 0 \tag{14}$$

or in other words we cannot at the same time have

$$a_2 = \pm is \quad \text{and} \quad a_1 = 0.$$

The equations for $\Gamma^{\sigma}_{\lambda K}$ and $\Gamma^{\sigma}_{L\kappa}$ respectively have the determinants:

$$\frac{(2s)^8}{(-s^2 + a_2^2)^4(s^2 + a_1^2)^8}\,[(s^2 + a_2^2)^2 + 4a_1^2 a_2^2]^2$$

which again yields no new condition.

If we introduce the covariant expressions (scalar densities)

$$\begin{aligned} I_1 &= |\, s_{ik} \,| \\ I_2 &= \tfrac{1}{4} e^{ijkl} e^{i'j'k'l'} s_{ii'} s_{jj'} a_{kk'} a_{ll'} \\ I_3 &= |\, a_{ik} \,|. \end{aligned}$$

Then we can sum up the conditions (10), (11), (13) and (14) as follows:

The necessary and sufficient conditions for the existence of a unique non-singular solution of equations (2) are

$$I_1 \neq 0 \tag{A}$$

$$g = I_1 + I_2 + I_3 \neq 0 \tag{B}$$

$$(I_1 - I_2)^2 + I_3 \neq 0 \tag{C}$$

(A) and (B) imply in the physically meaningful case the inequalities

$$|\, g_{ik} \,| < 0 \quad \text{and} \quad |\, s_{ik} \,| < 0$$

where the latter guarantees the existence of a non-degenerate "light cone" in every point. Equation (C) states that in no point can the two equations $I_1 = I_2$ and $I_3 = 0$ be satisfied simultaneously. In order that this be excluded it is *sufficient* that e.g. everywhere in space the antisymmetric field be restricted by the inequality

$$|\, I_1 \,| > |\, I_2 \,|$$

(| | stands here for the absolute values).

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