ACTS 4302

SOLUTION TO MIDTERM EXAM

Derivatives Markets, Chapters 9, 10, 11, 12, 18.
October 21, 2010 (Thurs)

Problem 1

You are given:

(i) The current exchange rate is 0.011\$/¥.
(ii) A four-year dollar-denominated European put option on yen with a strike price of \$0.008 sells for \$0.0005.
(iii) The continuously compounded risk-free interest rate on dollars is 3%.
(iv) The continuously compounded risk-free interest rate on yen is 1.5%.

Calculate the price of a four-year yen-denominated European put option on dollars with a strike price of ¥125.

Solution. By the Put Call Parity (PCP) for currency options with x_0 as a spot exchange rate:

$$C(x_0, K, T) - P(x_0, K, T) = x_0 e^{-r_f T} - K e^{-r_d T}$$
$$C = 0.011 \cdot e^{-0.015 \cdot 4} - 0.008 \cdot e^{-0.03 \cdot 4} + 0.0005 = 0.003764046$$

By the Call-Put relationship in "foreign" and "domestic" currency:

$$P_f\left(\frac{1}{x_0}, \frac{1}{K}, T\right) = \frac{1}{K} \cdot \frac{1}{x_0} C_d(x_0, K, T)$$

Hence, the price of a four-year yen-denominated European put option on dollars with a strike price of $\frac{1}{0.008}$ =¥125 is

$$P_{¥} = \frac{1}{0.008} \cdot \frac{1}{0.011} \cdot 0.003764046 = 42.77$$

□

Problem 2

You are given:

(i) C(K, T) denotes the current price of a K-strike T-year European call option on a nondividend-paying stock.
(ii) P(K, T) denotes the current price of a K-strike T-year European put option on the same stock.
(iii) S denotes the current price of the stock.
(iv) The continuously compounded risk-free interest rate is r.

Which of the following is (are) correct?

(I) $0 \leq C(50, T) - C(55, T) \leq 5e^{-rT}$
(II) $50e^{-rT} \leq P(45, T) - C(50, T) + S \leq 55e^{-rT}$
(III) $45e^{-rT} \leq P(45, T) - C(50, T) + S \leq 50e^{-rT}$

Solution. Since the maximum possible value of the difference between the two calls is:

$$\max(C(K_1) - C(K_2)) = e^{-rT}(K_2 - K_1)$$

using $K_1 = 50$ and $K_2 = 55$, we have:

$$C(50, T) - C(55, T) \leq 5e^{-rT}$$

Since by "direction property": if $K_1 \leq K_2$ then

$$C(K_1) \geq C(K_2)$$

again using $K_1 = 50$ and $K_2 = 55$, we have

$$C(50, T) - C(55, T) \geq 0$$

Therefore, I is true.
By the Put Call Parity for a nondividend paying stock:

$$C(S, K, T) - P(S, K, T) = S - Ke^{-rT} \Leftrightarrow P(S, K, T) - C(S, K, T) + S = Ke^{-rT}$$

For $K = 50$, the equality becomes:

$$P(S, 50, T) - C(S, 50, T) + S = 50e^{-rT}$$

Since by "direction property": if $K_1 \leq K_2$ then

$$P(K_1) \leq P(K_2)$$

using $K_1 = 45$ and $K_2 = 50$, we have:

$$P(45) \leq P(50) \Rightarrow P(S, 45, T) - C(S, 50, T) + S \leq 50e^{-rT}$$

On the other hand, for $K = 45$, the Put Call Parity becomes:

$$P(S, 45, T) - C(S, 45, T) + S = 45e^{-rT}$$

Since by "direction property": if $K_1 \leq K_2$ then

$$C(K_1) \geq C(K_2)$$

using $K_1 = 45$ and $K_2 = 50$, we have:

$$C(50) \leq C(45) \Rightarrow P(S, 45, T) - C(S, 50, T) + S \geq 45e^{-rT}$$

Therefore, III is true. Thus, II is not true.
□

Problem 3

Assume the Black-Scholes framework.
Eight months ago, an investor borrowed money at the risk-free interest rate to purchase a one-year 75-strike European call option on a nondividend-paying stock. At that time, the price of the call option was 8.
Today, the stock price is 85. The investor decides to close out all positions.
You are given:

(i) The continuously compounded risk-free interest rate is 5%.
(ii) The stock's volatility is 26%.

Calculate the eight-month holding profit.

Solution. 8 months after purchasing the option, the remaining time to expiration is 4 months. We are given: Eur. call, $t = \frac{1}{3}$, $S = 85$, $K = 75$, $\sigma = 0.26$, $r = 0.05$, $\delta = 0$.

$$C = Se^{-\delta t}N(d_1) - Ke^{-rt}N(d_2), \text{ where}$$

$$d_1 = \frac{\ln(S/K) + (r - \delta + \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}} = \frac{\ln\frac{85}{75} + \left(0.05 - 0 + 0.5 \cdot 0.26^2\right)\frac{1}{3}}{0.26\sqrt{\frac{1}{3}}} = \frac{0.1531}{0.1501} = 1.01989 \approx 1.02$$

$$N(d_1) = N(1.02) = 0.8461$$

$$d_2 = d_1 - \sigma\sqrt{t} = 1.01989 - 0.1501 = 0.8698 \approx 0.87$$

$$N(d_2) = N(0.87) = 0.8078$$

$$Ke^{-rt} = 75e^{-0.05\cdot\frac{1}{3}} = 73.7604$$

$$Se^{-\delta t} = 85$$

$$C = 85 \cdot 0.8461 - 73.7604 \cdot 0.8078 = 71.9185 - 59.5836 = 12.3349$$

Therefore, the holding profit is:

$$12.3349 - 8 \cdot e^{0.05 \cdot \frac{2}{3}} = 12.3349 - 8.2712 = 4.0637 \approx 4.06$$

□

Problem 4

Assume the Black-Scholes framework holds. Consider an option on a stock.
You are given the following information at time 0:

(i) The stock price is S(0), which is greater than 80.
(ii) The option price is 2.34.
(iii) The option delta is -0.181
(iv) The option gamma is 0.035.

The stock price changes to 86.00. Using the delta-gamma approximation, you find that the option price changes to 2.21.
Determine S(0).

Solution. Delta-gamma approximation

$$C(S_{t+h}) = C(S_t) + \Delta\epsilon + \frac{1}{2}\Gamma\epsilon^2 \Leftrightarrow C(S_{t+h}) - C(S_t) = \Delta\epsilon + \frac{1}{2}\Gamma\epsilon^2, \epsilon = S_{t+h} - S_t$$

Thus:

$$\begin{aligned}
&2.21 - 2.34 = -0.181\epsilon + 0.5 \cdot 0.035\epsilon^2 \\
&0.0175\epsilon^2 - 0.181\epsilon + 0.13 = 0 \\
&D = 0.181^2 - 4 \cdot 0.0175 \cdot 0.13 = 0.023661 = 0.1538^2 \\
&\epsilon_1 = \frac{0.181 + 0.1538}{2 \cdot 0.0175} = 9.5663 \text{ or } \epsilon_2 = \frac{0.181 - 0.1538}{2 \cdot 0.0175} = 0.7765
\end{aligned}$$

Since $S(0) > 80$, $\epsilon = 86 - S(0) < 86 - 80 = 6$. Therefore, $\epsilon = 0.7765$ and

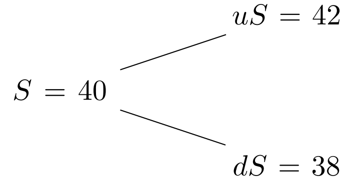
$$S(0) = 86 - 0.7765 = 85.2235$$

□

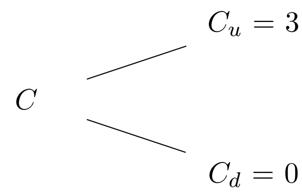
Problem 5

A nondividend paying stock's price is currently 40. It is known that at the end of one month the stock's price will be either 42 or 38. The continuously compounded risk-free rate is 0.08. Assume that the expected return on the stock is 10% as opposed to the risk-free rate. What is the true discount rate (rate of return) for the one-month European call option with a strike price of 39?

Solution.
We are given: $T = \frac{1}{12}, n = 1, S = 40, K = 39, S_u = 42, S_d = 38, \alpha = 0.1, r = 0.08, \delta = 0$. We need to find γ.
The stock tree:



The call tree:



Note that
$$u = \frac{42}{40} = 1.05 \text{ and } d = \frac{38}{40} = 0.95$$
Using equation that connects p and p^*:
$$e^{-\gamma h}\left(pC_u + (1-p)C_d\right) = e^{-rh}\left(p^*C_u + (1-p^*)C_d\right)$$
Since $C_d = 0$,
$$e^{-\gamma h}p = e^{-rh}p^* \Leftrightarrow \gamma = -\frac{1}{h}\ln\frac{e^{-rh}p^*}{p}$$
Using formulas for p and p^*:
$$p = \frac{e^{(\alpha-\delta)h} - d}{u-d} = \frac{e^{0.1\cdot\frac{1}{12}} - 0.95}{1.05-0.95} = 0.5837$$
$$p^* = \frac{e^{(r-\delta)h} - d}{u-d} = \frac{e^{0.08\cdot\frac{1}{12}} - 0.95}{0.1} = 0.5669$$
Therefore,
$$\gamma = -12\ln\frac{e^{-0.08\cdot\frac{1}{12}}0.5669}{0.5837} = 0.4303$$
□

Problem 6

A stock's price follows a lognormal model. You are given:
(i) The initial price is 90.
(ii) $\alpha = 0.12$.
(iii) $\delta = 0.06$.
(iv) $\sigma = 0.3$.
Construct a 92% confidence interval for the price of the stock at the end of five years.

Solution. The lognormal parameters m and v for the distribution of the stock price after five years are
$$m = (\alpha - \delta - 0.5\sigma^2)t = (0.12 - 0.06 - 0.5\cdot 0.3^2)5 = 0.015\cdot 5 = 0.075$$
$$v = \sigma\sqrt{t} = 0.3\sqrt{5} = 0.6708$$
If Z $\in N(m, v^2)$, then the $100(1-\alpha)\%$ confidence interval is defined to be
$$\left(m - z_{\alpha/2}v, m + z_{\alpha/2}v\right), \text{ where } z_{\alpha/2} \text{ is the number such that}$$
$$P\left[Z > z_{\alpha/2}\right] = \frac{\alpha}{2}$$
In our case $\alpha = 0.08 \Rightarrow \frac{\alpha}{2} = 0.04$ and
$$P\left[Z > z_{0.04}\right] = 0.04 \Leftrightarrow 1 - P\left[Z < z_{0.04}\right] = 0.04 \Leftrightarrow P\left[Z < z_{0.04}\right] = 0.96 \Leftrightarrow z_{0.04} = 1.75$$
The confidence interval is
$$\left(m - z_{\alpha/2}v, m + z_{\alpha/2}v\right) = (0.075 - 1.75\cdot 0.6708, 0.075 + 1.75\cdot 0.6708) = (-1.0989, 1.2489)$$

Exponentiating, we get

$$\left(e^{-1.0989}, e^{1.2489}\right) = (0.3332, 3.4865)$$

Multiplying by the initial stock price of 90, the answer is $(29.99, 313.79)$.
□

Problem 7

For a two-year 100-strike American call option on a stock is modeled with a 2-period binomial tree. You are given:

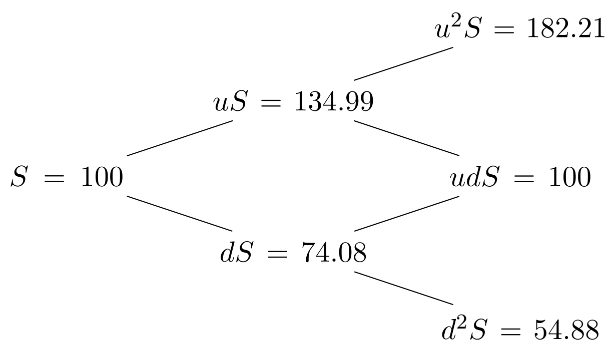
(i) The initial stock price is 100.
(ii) The tree is constructed based on forward prices.
(iii) The continuously compounded risk-free interest rate is 5%.
(iv) The stock's volatility is 30%.
(v) The stock pays dividends at the rate of 5%.

Determine the option premium.

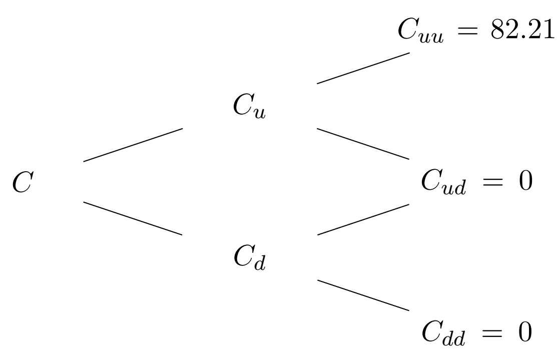
Solution. We are given: $T = 2, n = 2, r = 0.05, \gamma = -0.05, S = 100, K = 100, \sigma = 0.3$. We need to find C. Since the binomial tree is constructed using forward prices, we have:

$$u = e^{(r-\delta)h+\sigma\sqrt{h}} = e^{(0.05-0.05)\cdot 1+0.3\sqrt{1}} = e^{0.3} = 1.3499$$
$$d = e^{(r-\delta)h-\sigma\sqrt{h}} = e^{(0.05-0.05)\cdot 1-0.3\sqrt{1}} = e^{-0.3} = 0.7408$$

The stock tree:



The call tree:



Using the formula for risk neutral probability:

$$p^* = \frac{1}{1+e^{\sigma\sqrt{h}}} = \frac{1}{1+e^{0.3}} = 0.4256,\ 1 - p^* = 0.5744$$

We now use the formula for pricing an option with risk neutral probability:

$$C_u = e^{-rh}\left(p^* C_{uu} + (1-p^*)C_{ud}\right) = e^{-0.05\cdot 1}\cdot 0.4256 \cdot 82.21 = 33.28$$
$$C_d = e^{-rh}\left(p^* C_{ud} + (1-p^*)C_{dd}\right) = 0$$

Note that if the call is exercised at node u, the payoff will be $134.99 - 100 = 34.99$. This is greater than 33.28 obtained above, so it is optimal to exercise the call at that point and we'll use $C_u = 34.99$ in further calculations.

$$C = e^{-rh}\left(p^* C_u + (1-p^*)C_d\right) = e^{-0.05}\cdot 0.4256 \cdot 34.99 = 14.164$$

□