2

(a) $\int e^{-3x}\,dx = \int -\frac{1}{3} e^{u}\,du$

$u = -3x$ $\qquad = -\frac{1}{3} e^{u} + C$

$du = -3\,dx$

$\qquad = \frac{-1}{3} e^{-3x} + C$

$-\frac{1}{3} du = dx$

(b) $\int x e^{-3x}\,dx$ Integration by parts

$\left(\int u\,dv = uv - \int v\,du \right)$

$u = x$ $\qquad dv = e^{-3x}\,dx$

$du = dx$ $\qquad v = -\frac{1}{3} e^{-3x}$

$$\int x e^{-3x}\,dx = -\frac{1}{3} x e^{-3x} - \int -\frac{1}{3} e^{-3x}\,dx$$

$$= -\frac{1}{3} x e^{-3x} + \frac{1}{3} \int e^{-3x}\,dx$$

$$= -\frac{1}{3} x e^{-3x} + \frac{1}{3}\left(-\frac{1}{3} e^{-3x} \right) + C$$

$$= -\frac{1}{3} x e^{-3x} - \frac{1}{9} e^{-3x} + C.$$

(c) $\displaystyle\int \frac{1}{\sqrt{1-9x^2}}\,dx$ substitution

$$= \int \frac{1}{\sqrt{1-(3x)^2}}\,dx$$

$u = 3x$

$du = 3\,dx$

$\frac{1}{3}du = dx$

$$= \int \frac{\frac{1}{3}\,du}{\sqrt{1-u^2}} = \frac{1}{3}\arcsin u + C$$

$$= \frac{1}{3}\arcsin 3x + C$$

(d) $\displaystyle\int \frac{1}{1+4x^2}\,dx$ substitution

$$= \int \frac{1}{1+(2x)^2}\,dx$$

$u = 2x$

$du = 2\,dx$

$\frac{1}{2}du = dx$

$$= \int \frac{\frac{1}{2}\,du}{1+u^2} = \frac{1}{2}\arctan u + C$$

$$= \frac{1}{2}\arctan 2x + C$$

(e) $$\int \frac{1}{(x-1)\sqrt{x^2-2x-8}}\,dx$$ complete the square & substitution.

$$= \int \frac{1}{(x-1)\sqrt{x^2-2x+1-1-8}}\,dx$$

$$= \int \frac{1}{(x-1)\sqrt{(x-1)^2-9}}\,dx$$

$$= \int \frac{1}{(x-1)\sqrt{(x-1)^2-3^2}}\,dx$$

$u = x-1$

$du = dx$

$$= \int \frac{1}{u\sqrt{u^2-3^2}}\,du = \frac{1}{3}\operatorname{arcsec}\frac{|u|}{3} + C$$

$$= \frac{1}{3}\operatorname{arcsec}\frac{|x-1|}{3} + C.$$

(f) $$\int \frac{1}{x(\ln x)^3}\,dx = \int \frac{1}{u^3}\,du$$

$u = \ln x$

$du = \frac{1}{x}\,dx$

$$= \int u^{-3}\,du$$

$$= \frac{u^{-2}}{-2} + C$$

$$= -\frac{1}{2}(\ln x)^{-2} + C.$$

(g) $$\int \frac{\ln x}{x^2}\,dx$$

$u = \ln x$ $\qquad dv = \frac{1}{x^2}\,dx = x^{-2}\,dx$

$du = \frac{1}{x}\,dx$ $\qquad v = \frac{x^{-1}}{-1}$

$$\int \frac{\ln x}{x^2}\,dx = -\frac{\ln x}{x} - \int -\frac{1}{x^2}\,dx$$

$$= -\frac{\ln x}{x} + \int x^{-2}\,dx$$

$$= -\frac{\ln x}{x} + \frac{x^{-1}}{-1} + C$$

(h) $$\int x^2\sqrt{x-1}\,dx$$

$u = x^2$ $\qquad dv = (x-1)^{1/2}\,dx$

$du = 2x\,dx$ $\qquad v = \frac{(x-1)^{3/2}}{3/2}$

$$\int x^2\sqrt{x-1}\,dx = \frac{2}{3}x^2(x-1)^{3/2} - \int \frac{4}{3}x\,(x-1)^{3/2}\,dx$$

$$= \frac{2}{3}x^2(x-1)^{3/2} - \frac{4}{3}\underbrace{\int x\,(x-1)^{3/2}\,dx}_{(*)}$$

we calculate (*) separately

$$\int x(x-1)^{3/2}\,dx$$

$u = x$ $\qquad dv = (x-1)^{3/2}\,dx$

$du = dx$ $\qquad v = \dfrac{(x-1)^{5/2}}{5/2}$

$$\int x(x-1)^{3/2}dx = \frac{2}{5}x(x-1)^{5/2} - \int \frac{2}{5}(x-1)^{5/2}\,dx$$

$$= \frac{2}{5}x(x-1)^{5/2} - \frac{2}{5}\left(\frac{(x-1)^{7/2}}{7/2}\right) + C.$$

Plug this into (*) to get final answer.

(ii) $\int \cos^3 x \sin^2 x\,dx$

$$= \int \cos^2 x \sin^2 x \quad \cos x\,dx$$

$$= \int (1-\sin^2 x)\sin^2 x \quad \cos x\,dx$$

$$= \int (\sin^2 x - \sin^4 x)\cos x\,dx$$

$u = \sin x$

$du = \cos x\,dx$

$$= \int (u^2 - u^4)\,du = \frac{u^3}{3} - \frac{u^5}{5} + C = \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C$$

(j) $\int \sec^4 x \tan^3 x \, dx$

$= \int \sec^2 x \tan^3 x \quad \sec^2 x \, dx$

$= \int (\tan^2 x + 1) \tan^3 x \quad \sec^2 x \, dx$

$= \int (\tan^5 x + \tan^3 x) \quad \sec^2 x \, dx$

$u = \tan x$

$du = \sec^2 x \, dx$

$\vdots$

$= \dfrac{\tan^6 x}{6} + \dfrac{\tan^4 x}{4} + C$

(k) $\int \sin^3 x \, dx$

$= \int \sin^2 x \quad \sin x \, dx$

$= \int (1 - \cos^2 x) \quad \sin x \, dx$

$u = \cos x$

$du = -\sin x \, dx$

$-du = \sin x \, dx$

$= \int -(1 - u^2) \, du = -\left(u - \dfrac{u^3}{3}\right) + C$

$= -\cos x + \dfrac{\cos^3 x}{3} + C$

(l) $\int \sin^4 x \, dx$

$$\sin^2 x = \frac{1-\cos 2x}{2}$$

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

$$= \int (\sin^2 x)^2 \, dx$$

$$= \int \left(\frac{1-\cos 2x}{2}\right)^2 dx$$

$$= \int \frac{1}{4}\left(1 - 2\cos 2x + \cos^2 2x\right) dx$$

$$= \frac{1}{4}\int \left(1 - 2\cos 2x + \frac{1+\cos 4x}{2}\right) dx$$

$$= \frac{1}{4}\int \left(\frac{3}{2} - 2\cos 2x + \frac{1}{2}\cos 4x\right) dx$$

$$= \frac{1}{4}\left[\frac{3}{2}x - \sin 2x + \frac{1}{8}\sin 4x\right] + C$$

(m) $\int \cos^2 2x \, dx$

$$= \int \frac{1+\cos 4x}{2} \, dx$$

$$= \frac{1}{2}\int (1+\cos 4x) \, dx = \frac{1}{2}\left[x + \frac{1}{4}\sin 4x\right] + C$$

(n) $\displaystyle\int \frac{1}{(x^2+1)^{3/2}}\,dx$

$x = \tan\theta$

$dx = \sec^2\theta\, d\theta$

$$= \int \frac{1}{(\tan^2\theta + 1)^{3/2}} \sec^2\theta\, d\theta$$

$$= \int \frac{1}{(\sec^2\theta)^{3/2}} \sec^2\theta\, d\theta$$

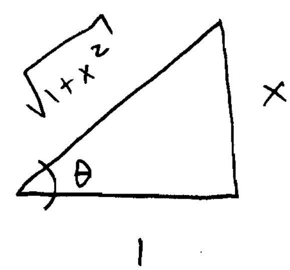
$$= \int \frac{1}{\sec\theta}\, d\theta$$

$$= \int \cos\theta\, d\theta$$

$$= \sin\theta + C$$

$$= \frac{x}{\sqrt{1+x^2}} + C.$$

$\tan\theta = x$



(o) $\displaystyle\int \frac{1}{(1-x^2)^{5/2}}\,dx$

$x = \sin\theta$

$dx = \cos\theta \; d\theta$

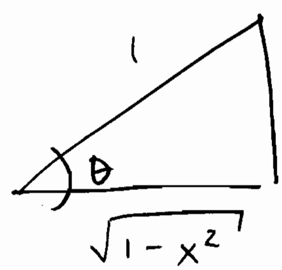
$= \displaystyle\int \frac{1}{(1-\sin^2\theta)^{5/2}} \cos\theta \; d\theta$

$= \displaystyle\int \frac{1}{(\cos^2\theta)^{5/2}} \cos\theta \; d\theta$

$= \displaystyle\int \frac{1}{\cos^4\theta} \; d\theta$

$\sin\theta = x$

$= \displaystyle\int \sec^4\theta \; d\theta$



$= \displaystyle\int \sec^2\theta \; \sec^2\theta \; d\theta$

$= \displaystyle\int (\tan^2\theta + 1) \; \sec^2\theta \; d\theta$

$(u = \tan\theta)$

$= \dfrac{\tan^3\theta}{3} + \theta + C$

$= \dfrac{\left(\dfrac{x}{\sqrt{1-x^2}}\right)^3}{3} + \arcsin x + C$.

(P) $\displaystyle \int \frac{x^3}{\sqrt{x^2+9}}\,dx = \int \frac{x^3}{\sqrt{x^2+3^2}}\,dx$

$x = 3\tan\theta$

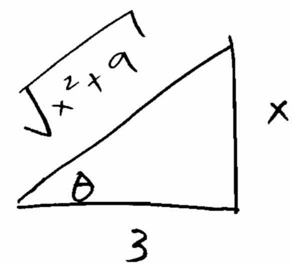
$dx = 3\sec^2\theta\,d\theta$

$$= \int \frac{27\tan^3\theta}{\sqrt{9\tan^2\theta + 9}}\quad 3\sec^2\theta\,d\theta$$

$$= \int \frac{27\tan^3\theta}{\sqrt{9(\underbrace{\tan^2\theta + 1}_{\sec^2\theta})}}\quad 3\sec^2\theta\,d\theta$$

$\tan\theta = \dfrac{x}{3}$

$$= \int \frac{27\tan^3\theta}{3\sec\theta}\cdot 3\sec^2\theta\,d\theta$$



$$= 27\int \tan^3\theta\,\sec\theta\,d\theta$$

$$= 27\int \tan^2\theta\quad \sec\theta\tan\theta\,d\theta$$

$$= 27\int (\sec^2\theta - 1)\quad \sec\theta\tan\theta\,d\theta \qquad (u = \sec\theta)$$

$$= 27\left(\frac{\sec^3\theta}{3} - \theta\right) + C$$

$$= 27\left(\frac{\left(\frac{\sqrt{x^2+9}}{3}\right)^3}{3} - \arctan\frac{x}{3}\right) + C$$

(8) $$\int \frac{x^2}{(1-x^2)^{3/2}}\,dx$$

$x = \sin\theta$

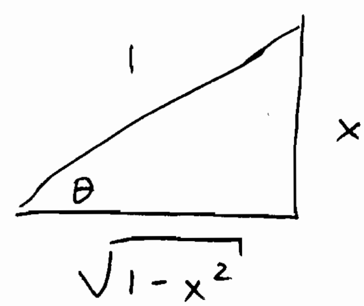
$dx = \cos\theta \, d\theta$

$$= \int \frac{\sin^2\theta}{(\underbrace{1-\sin^2\theta}_{\cos^2\theta})^{3/2}} \cos\theta \, d\theta$$

$$= \int \frac{\sin^2\theta}{\cos^3\theta} \cos\theta \, d\theta$$

$\sin\theta = \frac{x}{1}$

$$= \int \frac{\sin^2\theta}{\cos^2\theta} \, d\theta$$



$$= \int \tan^2\theta \, d\theta$$

$$= \int (\sec^2\theta - 1) \, d\theta$$

$$= \tan\theta - \theta + C$$

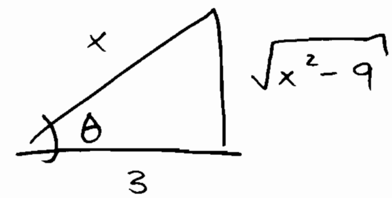
$$= \frac{x}{\sqrt{1-x^2}} - \arcsin x + C.$$

(r) $\displaystyle\int \frac{1}{\sqrt{x^2-9}}\,dx = \int \frac{1}{\sqrt{x^2-3^2}}\,dx$

$x = 3\sec\theta$

$\sec\theta = \frac{x}{3}$

$dx = 3\sec\theta\tan\theta\,d\theta$



$\displaystyle = \int \frac{1}{\sqrt{9\sec^2\theta-9}}\,3\sec\theta\tan\theta\,d\theta$

$\displaystyle = \int \frac{1}{\sqrt{9(\underbrace{\sec^2\theta-1}_{\tan^2\theta})}}\cdot 3\sec\theta\tan\theta\,d\theta$

$\displaystyle = \int \frac{1}{3\tan\theta}\,3\sec\theta\tan\theta\,d\theta$

$\displaystyle = \int \sec\theta\,d\theta = \ln|\sec\theta+\tan\theta| + C$

$\displaystyle = \ln\left|\frac{x}{3}+\frac{\sqrt{x^2-9}}{3}\right| + C.$

(s) $\displaystyle\int \frac{1}{(x^2-16)^{3/2}}\,dx = \int \frac{1}{(x^2-4^2)^{3/2}}\,dx$

$x = 4\sec\theta$

$dx = 4\sec\theta\tan\theta$

$\displaystyle = \int \frac{1}{(16\sec^2\theta-16)^{3/2}}\,4\sec\theta\tan\theta\,d\theta$

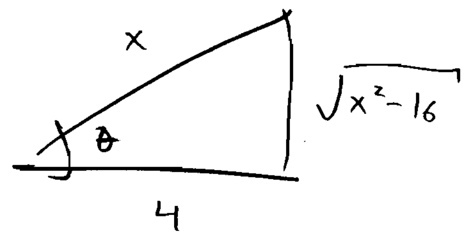
$$= \int \frac{1}{\left(16\left(\underbrace{\sec^2\theta - 1}_{\tan^2\theta}\right)\right)^{3/2}} \; 4\sec\theta\tan\theta \, d\theta$$

$$= \int \frac{1}{4^3 \tan^3\theta} \; 4 \cdot \sec\theta \tan\theta \; d\theta$$

$$= \frac{1}{16}\int \frac{\sec\theta}{\tan^2\theta} \, d\theta$$

$\sec\theta = \frac{x}{4}$

$$= \frac{1}{16}\int \frac{\frac{1}{\cos\theta}}{\frac{\sin^2\theta}{\cos^2\theta}} \, d\theta$$



$$= \frac{1}{16}\int \frac{\cos\theta}{\sin^2\theta} \, d\theta \qquad u = \sin\theta$$

$$= \frac{1}{16}\left(\frac{-1}{\sin\theta}\right) + C$$

$$= \frac{1}{16}\left(\frac{-1}{\frac{\sqrt{x^2-16}}{x}}\right) + C$$

(t)

$$\frac{4x^2+2x-1}{x^2(x+1)} = \frac{A}{x^2} + \frac{B}{x} + \frac{C}{x+1}$$

$$4x^2+2x-1 = A(x+1) + B(x)(x+1) + Cx^2$$

Plug $x=-1 \Rightarrow 1 = C$

Plug $x=0 \Rightarrow -1 = A$

$$4x^2+2x-1 = -1(x+1) + Bx(x+1) + x^2$$

Plug $x=1 \Rightarrow 5 = -2 + B(2) + 1$

$$B = 3$$

$$\int \frac{4x^2+2x-1}{x^2(x+1)}\,dx = \int \left(\frac{-1}{x^2} + \frac{3}{x} + \frac{1}{x+1} \right) dx$$

$$= \frac{1}{x} + 3\ln|x| + \ln|x+1| + C.$$

(u) $$\frac{x^2-1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$x^2-1 = A(x^2+1) + (Bx+C)x$$

Plug $x = 0$

$$-1 = A$$

$$x^2-1 = -1(x^2+1) + (Bx+C)x$$

$$= -x^2 - 1 + Bx^2 + Cx$$

$$= (B-1)x^2 + Cx - 1$$

Equating coefficients yields

$$B-1 = 1 \Rightarrow B = 2$$

$$C = 0$$

$$\int \frac{x^2-1}{x(x^2+1)}\,dx = \int \left(-\frac{1}{x} + \frac{2x}{x^2+1}\right) dx$$

$$= -\int \frac{1}{x}\,dx + \int \frac{2x}{x^2+1}\,dx$$

$(u = (x^2+1))$

$$= -\ln|x| + \ln|x^2+1| + C$$

3

(a) substitution $u = \operatorname{arcsec} x$

(b) $$\int_0^{\ln 2} \frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}} \, dx$$

$$= \int_0^{\ln 2} \frac{e^{2x} - \frac{1}{e^{2x}}}{e^{2x} + \frac{1}{e^{2x}}} \, dx$$

$$= \int_0^{\ln 2} \frac{\frac{e^{4x} - 1}{e^{2x}}}{\frac{e^{4x} + 1}{e^{2x}}} \, dx$$

$$= \int_0^{\ln 2} \frac{e^{4x} - 1}{e^{4x} + 1} \, dx$$

$u = e^{4x} + 1$

$du = e^{4x} \cdot 4 \, dx \implies dx = \frac{du}{4e^{4x}} = \frac{du}{4(u-1)}$

$$= \int_2^{17} \frac{u-2}{u} \cdot \frac{du}{4(u-1)}$$

$$= \int_2^{17} \frac{u-2}{4u(u-1)} \, du$$ Now partial fractions...

(c) Substitution $u = 1 - \ln x$

(d) $$\int_0^{\ln 2} \frac{2}{e^{-x}+1}\,dx$$

$$= \int_0^{\ln 2} \frac{2}{\frac{1}{e^x}+1}\,dx$$

$$= \int_0^{\ln 2} \frac{2}{\frac{1+e^x}{e^x}}\,dx$$

$$= \int_0^{\ln 2} \frac{2e^x}{1+e^x}\,dx$$

Now substitution $u = 1 + e^x$